XXI Congreso de Ecuaciones Diferenciales y Aplicaciones
XI Congreso de Matemática Aplicada
Ciudad Real, 21-25 septiembre 2009
(pp. 1–8)

On a pressure segregation scheme for the $3D$ Navier-Stokes equations and its relation to some pressure projection methods

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Keywords: Navier-Stokes Equations, pressure segregation, projection schemes, error estimates, first order time scheme, finite elements.

Abstract

A first-order linear fully discrete scheme for the incompressible time-dependent Navier-Stokes equations in three-dimensional domains is studied, which decouples the computations for velocity and pressure.

Using a C^0 finite element approximation verifying the *inf-sup* condition (the so-called *mini-element*), optimal error estimates are proved imposing the constraint $h \leq \alpha\, k$ where h and k are the mesh size and the time step respectively.

The main idea used here, is to define an artificial projection step in order to rewrite the scheme as a pressure incremental projection method, more easy to treat from the numerical analysis point of view.

1 Introduction

We consider the incompressible Navier-Stokes equations in a bounded domain $\Omega \subset I\!R^3$:

$$(P) \qquad \begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\,\Delta\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0 \;\text{ in } \Omega\times(0,T), \\ \mathbf{u} = 0 \;\text{ on } \partial\Omega\times(0,T), \quad \mathbf{u}_{|t=0} = \mathbf{u}_0 \;\text{ in } \Omega, \end{cases}$$

where $\mathbf{u}(\mathbf{x},t) \in I\!R^3$ is the fluid velocity at position $\mathbf{x}\in\Omega$ and time $t \in (0,T)$, $p(\mathbf{x},t) \in I\!R$ the pressure, $\nu > 0$ the viscosity (which is assumed constant) and $\mathbf{f}(\mathbf{x},t) \in I\!R^3$ the external force.

We consider a (regular) partition of $[0,T]$ of diameter $k = T/M$. If $u = (u^m)_{m=0}^M$ is a given vector with $u^m \in X$ (a Banach space), let us to introduce the following notation for discrete in time norms:

$$\|u\|_{l^2(X)} = \left(k \sum_{m=0}^{M} \|u^m\|_X^2 \right)^{1/2} \quad \text{and} \quad \|u\|_{l^\infty(X)} = \max_{m=0,\ldots,M} \|u^m\|_X$$

For simplicity, we will denote $H^1 = H^1(\Omega)$ etc., $L^2(H^1) = L^2(0,T;H^1)$ etc., and $\mathbf{H^1} = H^1(\Omega)^3$ etc.

The numerical analysis for the Navier-Stokes Problem (P) has received much attention in the last decades and many numerical schemes are now available. The main (numerical) difficulties of this problem are the coupling between the pressure p and the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ and the nonlinearity of the convective terms $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

The origin of so-called splitting in time methods is generally credited to the works of Chorin [2] and Temam [9]. They developed the well known *Chorin-Temam projection method*, which is a two step scheme, computing an intermediate velocity via a convection-diffusion problem and the second step is a free divergence $L^2(\Omega)$-projection step obtaining an end-of-step pair velocity-pressure. Afterwards, it was developed a modified projection scheme (called *incremental pressure or Van-Kan scheme*) where an explicit pressure term is added in the first step and a pressure correction term in the projection step. The main drawback of projection methods are that the end-of-step velocity does not satisfy the exact boundary conditions and the discrete pressure verifies "artificial" boundary conditions.

More recently, error estimates for projection methods have been obtained (see [7], [8] for time discrete schemes and see [4] for a fully discrete scheme). Basically, for the Chorin-Temam projection scheme, one has time error estimates of order $O(k^{1/2})$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ and of order $O(k)$ in $l^2(\mathbf{L}^2)$ for both velocities and of order $O(k^{1/2})$ in $l^2(L^2)$ for the pressure. For the incremental pressure scheme, the error estimates are improved to order $O(k)$ in $l^2(\mathbf{H}^1) \cap l^\infty(\mathbf{L}^2)$ for the intermediate velocity and order $O(k)$ in $l^2(L^2)$ for the pressure (this last estimate is proved for the linear problem) [7], [8]. In fact, these optimal estimates for the time discrete scheme, are also obtained in [4] for a fully discrete scheme by using a stable pair of finite element spaces and solving the projection step by means of a mixed velocity-pressure formulation, under the constraint $k^2 \le \alpha\, h$ in three-dimensional domains or $k^2 \le \alpha\,(1 + \log(h^{-1}))$ in two-dimensional cases.

By the contrary, in the present paper, we will obtain optimal error estimates under the completely different constraint $h \le \alpha\, k$ for a FEM scheme which decouple the computations of velocity and pressure. This result will be based on to rewrite the scheme as an incremental pressure projection method but solving the projection step by means of a Poisson problem for the pressure (more easy to treat from the computational point of view than the mixed velocity-pressure formulation given in [4]) and using the associated time discrete scheme as an intermediate problem (obtaining firstly the error estimates between the exact problem and this intermediate time scheme and afterwards the spatial error estimates between the intermediate time scheme and the fully discrete scheme). This argument has already been used in [5] for a different splitting scheme (with decomposition of viscosity) for Navier-Stokes equations.

The particular property that some projection methods can be rewritten as pressure segregation methods (decoupled the computations for velocity and pressure), was observed for instance in [7], and very recently, for the segregated scheme associated to the non-incremental projection method, the convergence and sub-optimal error estimates for the pressure (of order $O(k^{1/2}+h)$) have been obtained in [1], without imposing the inf-sup condition for the finite element spaces, but under the constraint $\alpha\, h^2 \le k \le \beta\, h^2$. In this paper, we study a pressure segregated scheme but related to the incremental projection method, obtaining optimal error estimates for the velocity and the pressure (of order $O(k+h)$) under the constraint $h \le \alpha\, k$ and imposing the inf-sup condition.

In the following, we will present the auxiliary time discrete scheme and the fully discrete scheme, jointly with the main results (in [6] the reader may see more details, in particular the proof of the results).

2 The auxiliary time discrete scheme

2.1 Description of the scheme

Given a (uniform) partition of the time interval $[0,T]$ with diameter $k = T/M$, $\{t_m = m\,k\}_{m=0}^M$, and $(\mathbf{f}^m)_{m=1}^M$ an approximation of $\mathbf{f}(t_m)$, we will define $(\mathbf{u}^m, p^m)_{m=1}^M$ an approximation of the solution $\{\mathbf{u}, p\}$ of (P) at the time $t = t_m$, by means of a two-step scheme splitting the nonlinearity $(\mathbf{u}\cdot\nabla)\mathbf{u}$ and the incompressibility condition $\nabla\cdot\mathbf{u} = 0$ into different steps (but keeping pressure terms in both steps): an explicit pressure term is introduced in the convection-diffusion problem for the velocity (step 1), with an implicit correction in the free-divergence projection step (step 2).

In the sequel, as usual, we will use the usual skew-symmetric form of the convective term $\langle C(\mathbf{u},\mathbf{v}),\mathbf{w}\rangle = c(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\nabla\mathbf{v},\mathbf{w}) - (\mathbf{u}\cdot\nabla\mathbf{w},\mathbf{v}))/2$. This trilinear form verifies

$$c(\mathbf{u},\mathbf{v},\mathbf{v}) = 0, \qquad \forall\,\mathbf{u}\in\mathbf{H}_0^1, \ \ \forall\,\mathbf{v}\in\mathbf{H}^1, \tag{1}$$

For simplicity, we take the viscosity constant $\nu = 1$.

The auxiliary time discrete scheme is defined as follows:

Initialization: Let $\widetilde{\mathbf{u}}^0 = \mathbf{u}(0)$. Let p^0 be given and to take $\mathbf{u}^0 = \widetilde{\mathbf{u}}^0$.

Sub-step 1 : Given $\mathbf{u}^m$, $\widetilde{\mathbf{u}}^m$ and p^m, to find $\widetilde{\mathbf{u}}^{m+1} : \Omega \to I\!R^3$ solution of

$$(S_1)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\widetilde{\mathbf{u}}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta\widetilde{\mathbf{u}}^{m+1} + \nabla p^m = \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Sub-step 2 : Given p^m and $\widetilde{\mathbf{u}}^{m+1}$, to find $\mathbf{u}^{m+1} : \Omega \to I\!R^3$ and $p^{m+1} : \Omega \to I\!R$ solving

$$(S_2)^{m+1} \qquad \begin{cases} \dfrac{1}{k}(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}) + \nabla\,(p^{m+1} - p^m) = 0 \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u}^{m+1} = 0 \quad \text{in } \Omega, \quad \mathbf{u}^{m+1}\cdot\mathbf{n}|_{\partial\Omega} = 0. \end{cases}$$

This step is a projection step. In fact, $\mathbf{u}^{m+1} = P_{\mathbf{H}}\widetilde{\mathbf{u}}^{m+1}$ where $P_{\mathbf{H}}$ is the L^2-projection onto $\mathbf{H}$, because $(\mathbf{u}^{m+1} - \widetilde{\mathbf{u}}^{m+1}, \mathbf{v}) = 0$ for all $\mathbf{u}\in\mathbf{H}$.

It is well known that Sub-step 2 is equivalent to the two following (decoupled) problems:

1. To find $p^{m+1} : \Omega \to I\!R$ such that

$$(S_2)_a^{m+1} \qquad \left\{ \begin{array}{l} k\,\Delta(p^{m+1} - p^m) = \nabla \cdot \widetilde{\mathbf{u}}^{m+1} \quad \text{in } \Omega \\ k\,\nabla(p^{m+1} - p^m) \cdot \mathbf{n}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega. \end{array} \right.$$

2. To find $\mathbf{u}^{m+1} : \Omega \to I\!R^3$ as

$$(S_2)_b^{m+1} \qquad \mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m) \quad \text{in } \Omega.$$

Notice that an initial pressure p^0 must be introduced as approximation of $p(0)$.

On the other hand, adding $(S_1)^{m+1}$ and $(S_2)^{m+1}$, we get the consistence relations:

$$(S_3)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{u}^{m+1} - \mathbf{u}^m) + C(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{u}}^{m+1}) - \Delta\widetilde{\mathbf{u}}^{m+1} + \nabla p^{m+1} = \mathbf{f}^{m+1} \quad \text{in } \Omega, \\ \quad \widetilde{\mathbf{u}}^{m+1}|_{\partial\Omega} = 0, \quad \nabla \cdot \mathbf{u}^{m+1} = 0 \quad \text{in } \Omega. \end{array} \right.$$

2.2 Differential problems verified by the errors

We introduce the following notations for the errors in $t = t_{m+1}$:

$$\widetilde{\mathbf{e}}^{m+1} = \mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}^{m+1}, \quad \mathbf{e}^{m+1} = \mathbf{u}(t_{m+1}) - \mathbf{u}^{m+1}, \quad e_p^{m+1} = p(t_{m+1}) - p^{m+1}.$$

Then, the errors verify the following problems:

$$(E_1)^{m+1} \quad \left\{ \begin{array}{l} \dfrac{1}{k}(\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m) - \Delta\widetilde{\mathbf{e}}^{m+1} + \nabla\left(e_p^m + k\,\delta_t p(t_{m+1})\right) = \mathcal{E}^{m+1} + \mathbf{NL}^{m+1} \quad \text{in } \Omega, \\ \widetilde{\mathbf{e}}^{m+1}|_{\partial\Omega} = \mathbf{0}, \end{array} \right.$$

where $\mathcal{E}^{m+1} = -\dfrac{1}{k}\displaystyle\int_{t_m}^{t_{m+1}} (t - t_m)\,\mathbf{u}_{tt}(t)\,dt - \left(\int_{t_m}^{t_{m+1}} \mathbf{u}_t \cdot \nabla\right) \mathbf{u}(t_{m+1}) := \mathcal{E}_1^{m+1} + \mathcal{E}_2^{m+1}$ is the consistency error, and $\mathbf{NL}^{m+1} = -C\Big(\widetilde{\mathbf{e}}^m, \mathbf{u}(t_{m+1})\Big) - C\Big(\widetilde{\mathbf{u}}^m, \widetilde{\mathbf{e}}^{m+1}\Big)$,

$$(E_2)^{m+1} \qquad \left\{ \begin{array}{l} \dfrac{1}{k}(\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}) + \nabla\Big(e_p^{m+1} - e_p^m - k\,\delta_t p(t_{m+1})\Big) = \mathbf{0} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{e}^{m+1} = 0 \quad \text{in } \Omega, \qquad \mathbf{e}^{m+1} \cdot \mathbf{n}|_{\partial\Omega} = \mathbf{0}. \end{array} \right.$$

Lemma 1 *(Continuous dependence of the errors) The following inequalities hold*

$$|\mathbf{e}^{m+1}| \le |\widetilde{\mathbf{e}}^{m+1}|, \qquad |\mathbf{e}^{m+1} - \widetilde{\mathbf{e}}^{m+1}| \le |\widetilde{\mathbf{e}}^{m+1} - \mathbf{e}^m|.$$

Moreover, there exists $C = C(\Omega) > 0$ such that $\|\mathbf{e}^{m+1}\| \le C\,\|\widetilde{\mathbf{e}}^{m+1}\|$.

2.3 Regularity hypotheses.

In the sequel, we will assume the following regularity hypothesis on Ω:

(H0) $\Omega \subset I\!R^3$ such that the Poisson-Dirichlet problem in Ω has $\mathbf{H}^2(\Omega)$ regularity.

In order to obtain the different error estimates, the following regularity hypotheses for the (unique) solution $(\mathbf{u}, p)$ of (P) will appear:

(H1) $\mathbf{u} \in L^\infty(\mathbf{H}^2 \cap \mathbf{V})$, $p_t \in L^2(H^1)$, $\mathbf{u}_t \in L^2(\mathbf{L}^2)$, $\mathbf{u}_{tt} \in L^2(\mathbf{H}^{-1})$

(H2) $p_{tt} \in L^2(H^1)$, $\mathbf{u}_t \in L^\infty(\mathbf{L}^3) \cap L^3(\mathbf{H}^1)$, $\mathbf{u}_{tt} \in L^2(\mathbf{L}^2)$, $\mathbf{u}_{ttt} \in L^2(\mathbf{H}^{-1})$

(H3) $\mathbf{u}_{tt} \in L^\infty(\mathbf{H}^{-1})$

Unfortunately, to obtain hypotheses **(H1)**-**(H3)** is necessary to assume that $\mathbf{u}_t(0) \in \mathbf{H}^1$, which implies a non local compatibility condition for the data $\mathbf{u}_0$ and $\mathbf{f}$.

2.4 $O(k)$-error estimates for the velocities

Theorem 2 *Under the hypotheses of* **(H1)** *and the constraint* $|\nabla e_p^0| \le C$*, the following error estimates hold:*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} \le C\,k \quad \text{and} \quad \|e_p^{m+1}\|_{l^\infty(H^1)} \le C. \qquad (2)$$

Lemma 3 *Under hypotheses of Theorem 2, one has* $\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C, \quad \forall m.$

From the l^∞ in time estimates of $\|\widetilde{\mathbf{e}}^{m+1}\|_{H^2} \le C$ and $\|e_p^{m+1}\| \le C$, $\forall m$, one also has

$$\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} \le C \quad \text{and} \quad \|p^{m+1}\| \le C \quad \forall m.$$

2.5 $O(k)$-error estimates for the pressure

First, we are going to obtain error estimates for discrete time derivative of velocity, that we denote as

$$\delta_t \mathbf{e}^{m+1} = \frac{\mathbf{e}^{m+1} - \mathbf{e}^m}{k}, \quad \delta_t \widetilde{\mathbf{e}}^{m+1} = \frac{\widetilde{\mathbf{e}}^{m+1} - \widetilde{\mathbf{e}}^m}{k}.$$

Afterwards, we will obtain the optimal $O(k)$ estimates for the pressure.

Making $\delta_t(E_1)^{m+1}$ and $\delta_t(E_2)^{m+1}$, one obtains ($\forall\, m \ge 1$),

$$(D_1)^{m+1} \quad \frac{\delta_t\widetilde{\mathbf{e}}^{m+1} - \delta_t \mathbf{e}^m}{k} - \Delta \delta_t \widetilde{\mathbf{e}}^{m+1} + \nabla(\delta_t e_p^m + k\,\delta_t\delta_t p(t_{m+1})) = \delta_t(\mathcal{E}^{m+1} + \mathbf{NL}^{m+1})$$

where $\delta_t\delta_t p(t_{m+1}) = (\delta_t p(t_{m+1}) - \delta_t p(t_m))/k$, and

$$(D_2)^{m+1} \qquad \frac{\delta_t \mathbf{e}^{m+1} - \delta_t\widetilde{\mathbf{e}}^{m+1}}{k} + \nabla(\delta_t e_p^{m+1} - \delta_t e_p^m - k\,\delta_t\delta_t p(t_{m+1})) = 0.$$

Theorem 4 *Assuming the hypotheses of Theorem 2,* (**H2**) *and the following constraint on the initial approximation* $|\delta_t \mathbf{e}^1| + |k\,\nabla \delta_t e_p^1| \le C\,k$*, then*

$$\|\delta_t \mathbf{e}^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} + \|\delta_t \widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(L^2)\cap l^2(H^1)} \le C\,k$$

Theorem 5 *Under hypothesis of Theorem 4 and* (**H3**)*, the following error estimates hold*

$$\|\widetilde{\mathbf{e}}^{m+1}\|_{l^\infty(H^1)} + \|e_p^{m+1}\|_{l^\infty(L^2)} \le C\,k.$$

3 Fully discrete scheme

3.1 Finite element approximation and fully discrete scheme

We consider a finite element approximation of the time discrete scheme given above. We restrict ourselves to the case where Ω is a $2D$ polygon or a $3D$ polyhedron satisfying the regularity hypothesis of (**H0**). We consider two finite element spaces $\mathbf{Y}_h \subset \mathbf{H}_0^1(\Omega)$ and $Q_h \subset H^1(\Omega) \cap L_0^2(\Omega)$ (globally continuous functions and locally polynomials of degree at least 1) associated to a regular family of triangulations $\mathcal{T}_h$ of the domain Ω with mesh size h. Moreover, we assume the "*inf-sup*" condition ([3]) for $(\mathbf{Y}_h, Q_h)$: there exists $\beta > 0$ independent of h such that, $\inf_{q_h \in Q_h\setminus\{0\}} \left(\sup_{\mathbf{v}_h \in \mathbf{Y}_h\setminus\{0\}} \frac{(q_h, \nabla\cdot \mathbf{v}_h)}{\|\mathbf{v}_h\|\,|q_h|} \right) \ge \beta$. We assume interpolation operators with the following properties:

- $I_h : \mathbf{L}^2 \to \mathbf{Y}_h$ such that $(\mathbf{u} - I_h\mathbf{u}, \nabla q_h) = 0,\ \forall\, q_h \in Q_h$ and satisfying the approximating properties:

$$\|\widetilde{\mathbf{u}} - I_h\widetilde{\mathbf{u}}\|_{H^{-1}} \le C\,h\,\|\widetilde{\mathbf{u}} - I_h\widetilde{\mathbf{u}}\|_{L^2} \quad \forall\,\widetilde{\mathbf{u}} \in \mathbf{L}^2(\Omega),$$

$$\|\widetilde{\mathbf{u}} - I_h\widetilde{\mathbf{u}}\|_{H^1} \le C\,h\,\|\widetilde{\mathbf{u}}\|_{H^2} \quad \forall\,\widetilde{\mathbf{u}} \in \mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega),$$

- $J_h : H^1 \to Q_h$ defined by $\Big(\nabla(J_h p - p), \nabla q_h\Big) = 0\ \forall\, q_h \in Q_h$, satisfying

$$|p - J_h p| \le C\,h\,\|p - J_h p\|_{H^1} \le C\,h\,\|p\|_{H^1} \quad \forall\, p \in H^1(\Omega)\cap L^2_0(\Omega).$$

For instance, we can consider the $I\!P_1$-bubble $\times$ $I\!P_1$ approximation to construct $\mathbf{Y}_h \times Q_h$. Now, following the equality $\mathbf{u}^{m+1} = \widetilde{\mathbf{u}}^{m+1} - k\,\nabla(p^{m+1} - p^m)$, we define the time-space interpolator operator $K_{h,k}\mathbf{u}^{m+1} \in \mathbf{Y}_h + \nabla Q_h$ by:

$$K_{h,k}\mathbf{u}^{m+1} = I_h\widetilde{\mathbf{u}}^{m+1} - k\,\nabla J_h(p^{m+1} - p^m). \tag{3}$$

Then, we can obtain

$$|\mathbf{u}^{m+1} - K_{h,k}\,\mathbf{u}^{m+1}| \le C\left(h\,\|\widetilde{\mathbf{u}}^{m+1}\|_{H^2} + k\,\|p^{m+1} - p^m\|\right) \le C(k+h) \quad \forall m.$$

Finally, the following constraint between the discrete parameters (k,h) will be assumed:
(H) $\quad h \le \alpha\,k \quad$ with $\alpha > 0$ a constant independent of k and h.

A generic step $m+1$ of the fully discrete scheme is defined as follows:

Sub-step 1: Given $(\widetilde{\mathbf{u}}^m_h, p^m_h) \in \mathbf{Y}_h \times Q_h$ and $\mathbf{u}^m_h \in \mathbf{Y}_h + \nabla Q_h$, to find $\widetilde{\mathbf{u}}^{m+1}_h \in \mathbf{Y}_h$ solving

$$\frac{1}{k}\Big(\widetilde{\mathbf{u}}^{m+1}_h - \mathbf{u}^m_h, \mathbf{v}_h\Big) + c\Big(\widetilde{\mathbf{u}}^m_h, \widetilde{\mathbf{u}}^{m+1}_h, \mathbf{v}_h\Big) + \Big(\nabla\,\widetilde{\mathbf{u}}^{m+1}_h, \nabla\,\mathbf{v}_h\Big) + \Big(\nabla p^m_h, \mathbf{v}_h\Big) = \Big(\mathbf{f}^{m+1}, \mathbf{v}_h\Big).$$

Sub-step 2: Given $(\widetilde{\mathbf{u}}^{m+1}_h, p^m_h) \in \mathbf{Y}_h \times Q_h$, to find $p^{m+1}_h \in Q_h$ solving

$$\Big(k\,\nabla(p^{m+1}_h - p^m_h), \nabla q_h\Big) = \Big(\widetilde{\mathbf{u}}^{m+1}_h, \nabla q_h\Big) \quad \forall q_h \in Q_h.$$

Now, we define $\mathbf{u}^{m+1}_h \in \mathbf{Y}_h + \nabla Q_h$ by

$$(S_2)^{m+1}_{b,h} \qquad \mathbf{u}^{m+1}_h = \widetilde{\mathbf{u}}^{m+1}_h - k\,\nabla(p^{m+1}_h - p^m_h).$$

It is very important that $\mathbf{u}^{m+1}_h$ verifies the L^2-orthogonality: $\Big(\mathbf{u}^{m+1}_h, \nabla q_h\Big) = 0\ \forall\, q_h \in Q_h$.

We introduce the end-of-step velocity $\mathbf{u}^m_h$ only to make the numerical analysis. This is not necessary for the practical implementation of this scheme, which can be realized as follows: Given $(p^{m-1}_h, \widetilde{\mathbf{u}}^m_h) \in Q_h \times \mathbf{Y}_h$.

(a) To find $p^m_h \in Q_h$ such that $\Big(k\,\nabla(p^m_h - p^{m-1}_h), \nabla q_h\Big) = \Big(\widetilde{\mathbf{u}}^m_h, \nabla q_h\Big) \quad \forall\, q_h \in Q_h$.

(b) To find $\widetilde{\mathbf{u}}_h^{m+1} \in \mathbf{Y}_h$ such that, $\forall\, \mathbf{v}_h \in \mathbf{Y}_h$,

$$\Big(\frac{\widetilde{\mathbf{u}}_h^{m+1} - \widetilde{\mathbf{u}}_h^m}{k}, \mathbf{v}_h\Big) + c\Big(\widetilde{\mathbf{u}}_h^m, \widetilde{\mathbf{u}}_h^{m+1}, \mathbf{v}_h\Big)$$
$$+\Big(\nabla\, \widetilde{\mathbf{u}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla(2p_h^m - p_h^{m-1}), \mathbf{v}_h\Big) = \Big(\mathbf{f}^{m+1}, \mathbf{v}_h\Big).$$

Then, the computation for pressure and velocity is decoupled (for this, the scheme is called a pressure segregation method). In fact, (a) is a Poisson problem for the pressure and (b) is a linear convection-diffusion problem for the velocity (which is also decoupled by components of $\widetilde{\mathbf{u}}_h^{m+1}$).

3.2 Problems related to the spatial discrete errors

We will present an error analysis for the fully discrete scheme $(\widetilde{\mathbf{u}}_h^{m+1}, \mathbf{u}_h^{m+1}, p_h^{m+1})$ as an approximation of the time discrete scheme $(\widetilde{\mathbf{u}}^{m+1}, \mathbf{u}^{m+1}, p^{m+1})$. We denote the errors as:

$$\mathbf{e}_d^{m+1} = \mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}, \qquad \widetilde{\mathbf{e}}_d^{m+1} = \widetilde{\mathbf{u}}^{m+1} - \widetilde{\mathbf{u}}_h^{m+1}, \quad e_{p,d}^{m+1} = p^{m+1} - p_h^{m+1}$$

These errors can be decomposed splitting the discrete part and the interpolation one:

$$\mathbf{e}_d^{m+1} = \mathbf{e}_h^{m+1} + \mathbf{e}_i^{m+1}, \qquad \widetilde{\mathbf{e}}_d^{m+1} = \widetilde{\mathbf{e}}_h^{m+1} + \widetilde{\mathbf{e}}_i^{m+1}, \qquad e_{p,d}^{m+1} = e_{p,h}^{m+1} + e_{p,i}^{m+1}$$

where $\mathbf{e}_i$ are the interpolation errors and $\mathbf{e}_h$ the space discrete errors. Then, one has

$$\left\{\begin{array}{l} \dfrac{1}{k}\Big(\widetilde{\mathbf{e}}_h^{m+1} - \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla\, \widetilde{\mathbf{e}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla e_{p,h}^m, \mathbf{v}_h\Big) = \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(e_i(\delta_t \widetilde{\mathbf{u}}^{m+1}), \mathbf{v}_h\Big) - \Big(\nabla \widetilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(\nabla(2\, e_{p,i}^m - e_{p,i}^{m-1}), \mathbf{v}_h\Big), \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h \end{array}\right.$$

$$\mathbf{e}_h^{m+1} = \widetilde{\mathbf{e}}_h^{m+1} - k\nabla(e_{p,h}^{m+1} - e_{p,h}^m).$$

3.3 $O(h)$ error estimates for $\widetilde{\mathbf{e}}_h^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and for $\mathbf{e}_h^{m+1}$ in $l^\infty(\mathbf{L}^2)$

Theorem 6 *We assume the hypotheses of Theorem 2,* $|\mathbf{e}_h^0| \le C\, h$ *and* $k|\nabla e_{p,h}^0| \le C\, h$. *Then, the following error estimates hold*

$$\|\widetilde{\mathbf{e}}_h^{m+1}\|^2_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|\mathbf{e}_h^{m+1}\|^2_{l^\infty(\mathbf{L}^2)} + \|k\, \nabla e_{p,h}^{m+1}\|^2_{l^\infty(L^2)} \le C\, h^2 \tag{4}$$

Corollary 7 *Assuming the hypotheses of Theorem 6,* $(\mathbf{H})$ *and* $\|\mathbf{u}_h^0\|^2_{W^{1,6}} \le C_0$, *one has*

$$\widetilde{\mathbf{u}}_h^{m+1} \textit{ is bounded in } l^\infty(\mathbf{W}^{1,6}). \tag{5}$$

3.4 $O(h)$ error estimates for $\delta_t \mathbf{e}_h^{m+1}$ in $l^\infty(\mathbf{L}^2)$, $\delta_t \widetilde{\mathbf{e}}_h^{m+1}$ in $l^\infty(\mathbf{L}^2) \cap l^2(\mathbf{H}^1)$ and $(\widetilde{\mathbf{e}}_h^{m+1}, e_{p,d}^{m+1})$ in $l^\infty(\mathbf{H}^1 \times L^2)$

By making $\delta_t(E_1)_h^{m+1}$ and $\delta_t(E_2)_h^{m+1}$, thanks to the choice of interpolation operators,

$$\left\{\begin{array}{l} \dfrac{1}{k}\Big(\delta_t \widetilde{\mathbf{e}}_h^{m+1} - \delta_t \mathbf{e}_h^m, \mathbf{v}_h\Big) + \Big(\nabla \delta_t \widetilde{\mathbf{e}}_h^{m+1}, \nabla\, \mathbf{v}_h\Big) + \Big(\nabla \delta_t e_{p,h}^m, \mathbf{v}_h\Big) = \delta_t\, \mathbf{NL}_h^{m+1}(\mathbf{v}_h) \\ -\Big(\mathbf{e}_i(\delta_t \delta_t \widetilde{\mathbf{u}}^{m+1}), \mathbf{v}_h\Big) - \Big(\nabla \delta_t \widetilde{\mathbf{e}}_i^{m+1}, \nabla \mathbf{v}_h\Big) - \Big(\nabla(2\, \delta_t e_{p,i}^m - \delta_t e_{p,i}^{m-1}), \mathbf{v}_h\Big) \quad \forall\, \mathbf{v}_h \in \mathbf{Y}_h \end{array}\right.$$

$$\delta_t \mathbf{e}_h^{m+1} = \delta_t \widetilde{\mathbf{e}}_h^{m+1} - k\nabla(\delta_t e_{p,h}^{m+1} - \delta_t e_{p,h}^m)$$

Theorem 8 *Under the hypotheses of Theorems 4 and 6, assuming the hypothesis for the first step of the scheme*

$$|\delta_t \mathbf{e}_h^1| + |k\, \nabla \delta_t e_{p,h}^1| \leq C\, h,$$

then

$$\|\delta_t \mathbf{e}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)} + \|\delta_t \widetilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(\mathbf{L}^2)\cap l^2(\mathbf{H}^1)} + \|k\, \delta_t \nabla e_{p,h}^{m+1}\|_{l^\infty(\mathbf{L}^2)} \leq C\, h. \tag{6}$$

Corollary 9 *Assuming hypotheses of Theorem 8, the following error estimates hold*

$$\|e_{p,h}^{m+1}\|_{l^\infty(L^2)} \leq C\, h \quad \text{and} \quad \|\widetilde{\mathbf{e}}_h^{m+1}\|_{l^\infty(H^1)} \leq C\, h.$$

Finally, combining Theorem 5 and Corollary 9, the the total error verifies

$$\|\mathbf{u}(t_{m+1}) - \widetilde{\mathbf{u}}_h^{m+1}\|_{l^\infty(H^1)} + \|p(t_{m+1}) - p_h^{m+1}\|_{l^\infty(L^2)} \leq C\,(k+h).$$

Acknowledgements

The first author has been partially supported by DGI-MEC (Spain), Grant MTM2006–07932 and Junta de Andalucía project P06-FQM-02373, and the second one by the research group FQM-315 of Junta de Andalucía.

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