XXI Congreso de Ecuaciones Diferenciales y Aplicaciones
XI Congreso de Matemática Aplicada
Ciudad Real, 21-25 septiembre 2009
(pp. 1–8)

On a Bernoulli-type problem with a Randon measure data depending on the own solution

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Palabras clave: Bernoulli problem, Non local problem, Radon measure data.

Resumen

Our aim is to study the existence and the numerical approach of solutions to a nonlinear Bernoulli-type free boundary problems with Radon measure data depending on the own solution in the one–dimensinal case. In particular we find a numerical solutions for some special cases.

1. Introduction and statement of the problem

We observed that most semilinear problems, of the form $-\Delta u(x) = F\left(x, u(x)\right),\ x \in \Omega$ (with Ω a given open bounded set in $\mathbb{R}^N$) and some boundary conditions on $\partial\Omega$, have been intensively studied in the literature when F is a given function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$. Nevertheless, some relevant models in Physics can be expressed as $-\Delta u(x) = \mu(x,\ u)$ in $\mathcal{D}'(\Omega)$, where $\mu(x,\ u)$ is a *Radon measure* depending on x but also on the own solution u. One example of the above mentioned problems, involving u–dependent measures, corresponds to the so called *interior Bernoulli problem* on Ω. For some special cases, this problem can be stated as follows: to find a function $u : \bar{\Omega} \subset \mathbb{R}^N \to \mathbb{R}$ and a subset $A \subset\subset \Omega$ such that

$$(\mathcal{B}) \begin{cases} -\Delta_p u \doteq \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) & = & 0 & \text{in } \Omega\backslash\overline{A}, \\ u & = & 0 & \text{on } \partial\Omega, \\ u & = & 1 & \text{on } \partial A, \\ -|\nabla u|^{p-2}\nabla u\cdot\overrightarrow{n} & = & q & \text{on } \partial A, \end{cases}$$

where q is a given continuous function on $\overline{\Omega}$, $N \geqslant 2$ and $1 < p < +\infty$.

This problem arises, for instance, *in ideal fluid dynamics*: (for $p = 2, N = 2$), in the study of a inviscid incompressible irrotational horizontal flow in stationary regime. The function u is the stream function, i.e. the vector velocity $\mathbf{v}$ is given trough its stream function u by $\mathbf{v}=(\partial_2 u, -\partial_1 u)$. The incompressibility of the flow implies that u is an harmonic function. If we assume that the fluid circulate in Ω around a bubble of air A (of unknown location in Ω). Since $\partial\Omega$ and ∂A are streamlines, after a normalization, we can assume that $u \equiv 1$ on A and $u = 0$ on $\partial\Omega$. Moreover, the (Daniel) *Bernoulli principle* holds on ∂A leading to $\frac{1}{2}|\mathbf{v}|^2 + \frac{p}{\rho} + gz = const$ and so $|\nabla u|$ must be constant on ∂A. For a mathematical treatment of the problem see, for instance, [2], [1], [3], [9] and the survey Flucher and Rumpf [7] where a long list of references, as well as some other applications to electrolytic drilling and galvanization, can be found. Some other references, dealing with the case $p \neq 2$), are given in Henrot [10]. Similar formulation comes from the magnetic confinement of a plasma in the ideal nuclear fusion. Particularly the so-called *sharp problem of the ideal MHD* is modeled by the Grad–Safranov's equation $-\mathcal{L}u = a(x)\,F(u) + F(u)\frac{dF}{du}(u) + b(x)\,\frac{dP}{du}(u)$ in Ω, where u is the stream function of the averaged magnetic field $\hat{\mathbf{B}} = (\partial_2 u, -\partial_1 u)$, $\mathcal{L}u$ is a second order elliptic operator, $F(u)$ is the third covariant component of $\mathbf{B}$ (F is an unknown real function) and $P(u)$ is the pressure which here is assumed to be $P(u) \equiv 1$ in the plasma region ($\{x \in \Omega : u(x) > 0\}$) and $P(u) \equiv 0$ in the vacuum region ($\{x \in \Omega : u(x) < 0\}$) and it is a constitutive law (see Bruno and Laurence [5] and Friedman and Liu [8]).

Now, we reformulated problem $(\mathcal{B})$ and we can prove from Strong Maximum Principle that

Theorem 1 *Bernoulli's problem $(\mathcal{B})$ has a solution if and only if there exists a solution u of problem*

$$(\mathcal{B}_1)\left\{\begin{array}{l} \text{Look for } u \in C\left(\bar{\Omega}\right) \cap W_0^{1,p}(\Omega) \text{ such that} \\ \displaystyle\int_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla\varphi(x)\,dx = u_*(0)\,|u_*(0)\,|^{p-2}\int_{\partial\{u=u_*(0)\}} q\varphi dH_{N-1}, \\ \qquad\qquad \forall\varphi \in C\left(\bar{\Omega}\right) \cap W_0^{1,p}(\Omega),\ u_*(0) \neq 0, \end{array}\right.$$

where u_* is the nondecreasing rearrangement *and $u_*(0) = ess\sup_\Omega u$ (see [11]), and by $\partial\{u = u_*(0)\}$ we denote the boundary of the set $\{x \in \Omega : u(x) = u_*(0)\}$. Here, H_m denotes the m–dimensional Hausdorff measure.*

So, for any $u \in C\left(\bar{\Omega}\right) \cap W_0^{1,p}(\Omega)$, if we define the operator $\mathcal{A}$ and the measure μ as

$$\langle \mathcal{A}u, \varphi\rangle \doteq \int_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla\varphi dx = \langle \mu(\cdot, u), \varphi\rangle$$
$$\text{with } \langle \mu(\cdot, u), \varphi\rangle \doteq \int_{\partial(u^{-1}(1))} q\varphi dH_{N-1} \ \text{ for all } \varphi \in C\left(\bar{\Omega}\right) \cap W_0^{1,p}(\Omega);$$

where we denote by $\langle\cdot,\ \cdot\rangle$ the duality product between some functional space V and its dual space V', the Bernoulli problem $(\mathcal{B})$ is a particular case of the following problem $(\mathcal{P}_\mu)$:

Given a reflexive Banach space $(V, ||\cdot||)$ of dual V', we assume that $(V, ||\cdot||) \hookrightarrow \left(C(\overline{\Omega}), |\cdot|_\infty\right)$ with compact embedding. We consider an abstract *pseudomonotone* operator $\mathcal{A} : V \to V'$ (see [4]). Now the problem is to find $u \in V(\Omega)$ with a non–void set $\partial\left(u^{-1}(1)\right)$ (:=boundary of set $\{x \in \Omega : u(x) = 1\}$) and find a non-negative bounded Radon measure μ whose support is non–void and included in the set $\partial\left(u^{-1}(1)\right)$ such that

$$(\mathcal{P}_\mu) \qquad \langle \mathcal{A}u, \varphi\rangle = \langle \mu(x, u), \varphi\rangle \qquad \forall\varphi \in V(\Omega)\ .$$

The measure $\mu(x,u)$ will be defined by: for a given $q \in C(\overline{\Omega})$ and for all $\varphi \in C(\overline{\Omega})$

$$\langle \mu(x,u), \varphi\rangle = \int_{\partial(u^{-1}(1))} q(y)\varphi(y)dH_{N-1}(y), \text{ if } N \geq 2 \text{ and } \langle \mu(x,u), \varphi\rangle = \sum_{\partial(u^{-1}(1))} q(y)\varphi(y), \text{ for } N=1.$$

Example 1 This is satisfied for $A(v) = \Delta_p v$ and for $A(v) = \Delta(|\Delta v|^{p-2}\Delta v) + b(x)\phi(v(x))$, with ϕ a continuous bounded enough function and $V = W_0^{2,p}(\Omega)$, $p > \frac{N}{2}$.

In this more general situation, the existence of solution was proved by Díaz-Padial-Rakotoson in [6] by using a family of approximate quasilinear (non–singular) non–monotone equations $\mathcal{A}u_n = qF_n(u_n)$ for suitable functions F_n. They introduced an approximate sequence of functionals for that family of equations. They proved the existence of a critical point for each of them (verifying a Palais–Smale condition–type) and from uniform a priori estimates, passing to the limit, they prove the existence of solutions. This existence result is the following:

Theorem 2 ([6]) *We consider continuous positive function $q \in C(\bar{\Omega})$ and an abstract bounded, strongly-weakly continuous and* pseudomonotone *operator $\mathcal{A} : V \to V'$ such that:*

i) *If $\mathcal{A}(v) = 0$ for some $v \in V$, then $v = 0$.*

ii) *There is a Gâteaux differentiable function $J : V \to \mathbb{R}$ such that:*

a) $\langle \mathcal{A}u, \varphi\rangle = \langle J'(u), \varphi\rangle \left(:= \lim_{t\to 0} \frac{J(u+t\varphi)-J(u)}{t}\right)$, $\forall \varphi \in V$, $\forall u \in V$.

b) *J is coercive in the following sense: there are some nonnegative constants α_i, $i = 0, 1, 2, 3$ such that, $\forall v \in V$, $\alpha_0\|v\| - \alpha_1 \leqslant J(v) \leqslant \alpha_2\|v\|^p + \alpha_3$ for some $p \geqslant 1$, and with $\alpha_0 K > 2\alpha_1 + \alpha_3$ (K given by $K = \inf_{|v|_\infty=1} ||v|| > 0$).*

c) $J(0) \leqslant 0$.

Then there exists (u,μ) solution of $(\mathcal{P}_\mu)$ with $u \in V$ and μ a nonnegative bounded Radon measure *whose support is non void and contained in $\partial(u^{-1}(1))$ such that:*

$$\langle \mathcal{A}u, \varphi\rangle = \int_{\partial(u^{-1}(1))} q(x)\varphi(x)d\mu(x) \ \forall \varphi \in V.$$

Moreover $\langle \mathcal{A}u, u\rangle > 0$.

The aim of this lecture is to look for the complete identification of Radon measure μ in the one–dimensional case and to show some geometry properties of u and μ depending of the symmetry of the elliptic operator. We will show an explicit and numerical solution for some special cases and we will show that we can not expect to have uniqueness of solution (u,μ) of problem $(\mathcal{P}_\mu)$.

2. The complete identification of μ

In this section, $\Omega =]0,1[$. We need the following assumption, typical of differential operators without zeroth order terms:

$$\left\{\begin{array}{l}\text{For any } [a,b] \subset]0,1[\text{ and any } v \in V \text{ such that } v(a)=v(b) \text{ and } \langle \mathcal{A}v, \varphi\rangle = 0,\\ \forall\varphi \in C_c^\infty[a,b], \text{ then, necessary, } v(x)=v(a) \text{ for all } x\in[a,b].\end{array}\right. \quad (1)$$

Example 2 *Given* $1 < p_1 \leq p_2 \leq \ldots \leq p_m$, $\alpha_i \in L^\infty(0,1)$, $i = 1,\ldots,m$, $\alpha_i(x) \geq \alpha_0 > 0$ *a.e., let* $\mathcal{A}v = -\sum_{i=1}^m \left(\alpha_i \left|v'\right|^{p_i-2} v'\right)'$ *and* $V = \left\{v \in W^{1,p_m}(0,1) : v(0) = 0\right\}$ *or* $V = W_0^{1,p_m}(0,1)$. *Then* $\mathcal{A}$ *satisfies the property (1).*

Proposition 1 *Assume* $\mathcal{A}$ *as in Theorem 2 and verifying (1). Then for any solution* u *of problem* $(\mathcal{P}_\mu)$*, there* exist two points $a_0, a_1 \in]0,1[$ *such that* $\{x \in \Omega : u(x) = 1\} = [a_0, a_1]$. *In particular* $\partial\left(u^{-1}(1)\right) = \{a_0, a_1\}$ *and there are* two nonnegative constants $\overline{\lambda_0}$, $\overline{\lambda_1}$, *with* $\overline{\lambda_0} + \overline{\lambda_1} > 0$ *and such that* $\mu = \overline{\lambda_0}\delta_{a_0} + \overline{\lambda_1}\delta_{a_1}$.

Proof. Let $a_0 = \text{mín}\{x \in]0,1[: u(x) = 1\}, a_1 = \text{máx}\{x \in]0,1[: u(x) = 1\}$. Let us show that $]a_0, a_1[\cap \{x : u(x) < 1\} = \emptyset$. If $a_0 = a_1$ then, there is nothing to be proved. If $a_0 < a_1$, we suppose that $]a_0, a_1[\cap \{u < 1\} \neq \emptyset$. Let $x_0 \in]a_0, a_1[$ be such that $u(x_0) < 1$. Let us denote by $I(x_0)$ the biggest interval containing x_0 and $\overset{\circ}{I}(x_0) \subset]a_0, a_1[\cap \{u < 1\}$. Then, on the $\partial\overset{\circ}{I}(x_0)$ we have $u = 1$. Thus one has that $\langle \mathcal{A}u, \varphi\rangle = 0$, $\forall\varphi \in C_c^\infty(\overset{\circ}{I}(x_0))$ and $u(x) = 1$, $x \in \partial\overset{\circ}{I}(x_0)$. From assumption (1), we deduce that $u = 1$ on $I(x_0)$ which is *absurd*, since $u(x_0) < 1$. This shows that $[a_0, a_1] \subset \{u \geq 1\}$. A similar argument shows that $]a_0, a_1[\cap \{u > 1\} = \emptyset$. Thus, we have $[a_0, a_1] \subset \{u = 1\}$ and by the definition of a_0, a_1 we have $\{u = 1\} \subset [a_0, a_1]$ and thus $\{u = 1\} = [a_0, a_1]$. This implies $\partial\{u = 1\} = \{a_0, a_1\}$. Finally, since $\mu \geq 0$ and $\text{supp}(\mu) \subset \partial(u^{-1}(1))$, we deduce that there exists $\overline{\lambda_0} \geq 0$, $\overline{\lambda_1} \geq 0$ such that $\mu = \overline{\lambda_0}\delta_{a_0} + \overline{\lambda_1}\delta_{a_1}$. The fact that $\mu \not\equiv 0$ implies that $\overline{\lambda_0} + \overline{\lambda_1} > 0$. □

As we have observed, the shape of μ depends on $(\mathcal{A}, V)$. Here is an example where $a_1 = 1 - a_0$, $\overline{\lambda_0} = \overline{\lambda_1}$. We define first the set of symmetric functions and operators:

Definition 1 *We set* $L_s^p(0,1) = \{v \in L^p(0,1),\ v(x) = v(1-x), a.e\}$ *and* $C_s[0,1] = \{v \in C[0,1],\ v(x) = v(1-x)\ \forall x\}$ *and we define the set* $V_s = V \cap C_s[0,1]$. *The elements of* $L_s^p(0,1)$ *are called* symmetric functions. *An operator* $\mathcal{A} : V \to V'$ *is called* symmetric *if* $\forall\varphi \in V$, $\forall v \in V_s$, $< \mathcal{A}v, \varphi > = < \mathcal{A}v, \varphi_s >$ *where* $\varphi_s(x) = \frac{\varphi(x)+\varphi(1-x)}{2}$ *and* $\varphi_s \in V_s$.

Remark 1 It is easy to see that if $\alpha_i \in L_s^\infty(0,\ 1)$ the operator $\mathcal{A}$ defined in Example 2 is a symmetric operator on the space $V_s = W_0^{1,p_m}(0,\ 1) \cap C_s[0,\ 1]$.

According to the results of Theorem 2 and Proposition 1, we have the following

Proposition 2 *Assume* $\mathcal{A}$ *as in Theorem 2 and verifying (1), and* $q \in C_s[0,\ 1]$ *with* $q > 0$*, then there exist* $u \in V_s$, $\overline{\lambda_0} \geqslant 0$, $\overline{\lambda_1} \geqslant 0$ *and* $a_0 \in [0,\ \frac{1}{2}]$ *such that*

$$\mathcal{A}u = q(a_0)(\overline{\lambda_0}\delta_{a_0} + \overline{\lambda_1}\delta_{1-a_0}) \text{ in } V_s'.$$

Moreover, if $\mathcal{A}$ *is symmetric then* $\mu = \frac{\langle \mathcal{A}u, u\rangle}{2q(a_0)}(\delta_{a_0} + \delta_{1-a_0})$.

Proof. Since V_s is a reflexive Banach space (since it is a closed subspace of V), applying Theorem 2 and all the above results, we get the existence of $u \in V_s$, $0 \leqslant a_0 \leqslant a_1 \leqslant 1$, $a_i \in \partial\,(u^{-1}(1))$ and $\overline{\lambda_0} \geqslant 0$, $\overline{\lambda_1} \geqslant 0$ such that $\mathcal{A}u = q(a_0)\overline{\lambda_0}\delta_{a_0} + \overline{\lambda_1}q(a_1)\delta_{a_1}$. Since u is symmetric $a_1 = 1 - a_0$ and then $q(a_0) = q(a_1)$, $0 \leqslant a_0 \leqslant \frac{1}{2}$. Finally, if $\mathcal{A}$ is symmetric then $\langle \mathcal{A}u, \varphi\rangle = \langle \mathcal{A}u, \varphi_s\rangle = q(a_0)(\overline{\lambda_0} + \overline{\lambda_1})\left(\frac{\varphi(a_0)+\varphi(1-a_0)}{2}\right)$ for any $\varphi \in V$. In particular if $\varphi = u$ then

$$\langle \mathcal{A}u, u\rangle = q(a_0)(\overline{\lambda_0} + \overline{\lambda_1}).$$

Thus $\mathcal{A}u = q(a_0)\frac{\langle \mathcal{A}u,u\rangle}{2q(a_0)}(\delta_{a_0} + \delta_{1-a_0})$. □

Remark 2 The result of Proposition 2 shows that the problem can be interpreted as a nonlinear problem of eigenvalue type.

The above identification allows to show that, in general, we can not expect to have *uniqueness* of solutions $(u,\ \mu)$ of problem $(\mathcal{P}_\mu)$. Indeed, when q is a positive constant the above arguments lead to the explicit construction a family of solutions:

Proposition 3 *Assume that q is positive constant. Then for any $\lambda > \frac{2}{q}$ the pair $(u_\lambda,\ \mu_\lambda)$ is a solution of the special problem $(\mathcal{P}_\mu)$*

$$\int_0^1 u'\varphi' dx = \int_{\partial\,(u^{-1}(1))} q\varphi d\mu, \quad \forall\varphi \in H_0^1(0,1),\ u \in H_0^1(0,1), \qquad (-u'' = \mu)$$

where u_λ is given by

$$u_\lambda(x) = \begin{cases} \lambda q x & if\ 0 \leqslant x \leqslant \frac{1}{\lambda q}, \\ 1 & if\ \frac{1}{\lambda q} \leqslant x \leqslant 1 - \frac{1}{\lambda q}, \\ \lambda q(1-x) & if\ 1 - \frac{1}{\lambda q} \leqslant x \leqslant 1, \end{cases}$$

and μ_λ is given by $\mu_\lambda = \lambda(\delta_{a_0} + \delta_{1-a_0})$ with $a_0 = \frac{1}{\lambda q}$. On the other hand, u_λ is the unique solution of

$$\int_0^1 u'\varphi' dx = \lambda q \sum_{a_i \in \partial\,(u^{-1}(1))} \varphi(a_i), \quad \forall\varphi \in H_0^1(0,1),\ u \in H_0^1(0,1).$$

It is possible to get the uniqueness of solution $(u,\ \mu)$ of problem $(\mathcal{P}_\mu)$, at least under the simple formulation given in the above proposition, once we add some extra condition, for instance, by prescribing the value of $\int_0^1 u'^2(x)dx$ in a right way:

Proposition 4 *Let $\mathcal{A}v = -v''$ and $\lambda > \frac{2}{q}$, then, the problem*

$$(\mathcal{P}_\lambda) \begin{cases} \text{find } u \in H_0^1(0,1) \text{ and a measure } \mu \geqslant 0 \text{ satisfying that} \\ -u'' = q\mu,\ \text{support}(\mu) \subset \partial(u^{-1}(1)),\ \frac{1}{2q}\int_0^1 u'^2(x)dx = \lambda, \end{cases}$$

possesses a unique solution. *This shows that the problem can be interpreted as a* nonlinear eigenvalue problem.

2.1. An explicit solution for an especial case of nonlinear Bernoulli problem

Given $\Omega =]0,1[$, a positive constant q, the problem is to find a regular function u and real number $0 < a_0 < a_1 < 1$ ($A =]a_0, a_1[\subset\subset \Omega$) such that

$$-\left(b(x)\left|u'(x)\right|^{p-2} u'(x)\right)' = 0 \qquad \text{in } \Omega \setminus \bar{A} =]0, a_0[\cup]a_1, 1[\tag{2}$$
$$u(x) = 0 \qquad \text{in } \partial\Omega = \{0, 1\},$$
$$u(x) = 1 \ \text{ in } \partial A = \{a_0, a_1\}, \quad \frac{\partial u}{\partial \bar{n}} = q \ \text{ in } \partial A \qquad \left(u'(a_0) = q,\, u'(a_1) = -q\right).$$

We assume that b is a continuous function in Ω, strictly positive and bounded: $0 < b_0 \leq b(x) \leq b_1$. We assume that for any $\varepsilon \in]0,1[$ it is $\frac{1}{|q|} < \left(\frac{b_0}{b_1}\right)^{\frac{1}{p-1}} \text{mín}\{\varepsilon, 1-\varepsilon\}$ and in particular if $\varepsilon = \frac{1}{2}$,

$$\frac{1}{|q|} < \frac{1}{2}\left(\frac{b_0}{b_1}\right)^{\frac{1}{p-1}} \tag{3}$$

First, we begin looking for a_0*:* integrating (2) between $x \in]0, a_0[$ and a_0, and assuming $u'(x) \geq 0$, we obtain that $u'(x) = |q|\left(\frac{b(a_0)}{b(x)}\right)^{\frac{1}{p-1}}$. Integrating again between 0 and $x \in]0, a_0[$,

$$u(x) = \int_0^x |q|\left(\frac{b(a_0)}{b(x)}\right)^{\frac{1}{p-1}} dx.$$

Since $u(a_0) = 1$, then $\frac{1}{|q|} = \int_0^{a_0}\left(\frac{b(a_0)}{b(x)}\right)^{\frac{1}{p-1}} dx$. We will prove that there exists $a_0 \in]0, 1-\varepsilon[$ solution of this equations. We define the continuous function

$$h(a) = 1 - |q|\int_0^a \left(\frac{b(a)}{b(x)}\right)^{\frac{1}{p-1}} dx$$

then, a_0 will belong to $\Gamma := \{a \in]0,1[: h(a) = 0\}$. We prove that $\Gamma \neq \emptyset$. Since $h(0) = 1 > 0$ and $h(1-\varepsilon) = 1 - |q|\int_0^{1-\varepsilon}\left(\frac{b(1-\varepsilon)}{b(x)}\right)^{\frac{1}{p-1}} dx < 0$ (from the assumption (3) of b), we can apply Bolzano's Theorem to obtain that there exists $a_0 \in]0, 1-\varepsilon[$ (if, moreover, b is nondecreasing, then $\exists! a \in]0,1[$ with $h(a) = 0$) such that the function

$$u(x) = \int_0^x |q|\left(\frac{b(a_0)}{b(x)}\right)^{\frac{1}{p-1}} dx \text{ for any } x \in]0, a_0[,\ u(x) = 1 \text{ for any } x \in [a_0, a_1]$$

verifies the problem (2) restricted to $\Omega_0 =]0, a_0[$.
Second, to look for a_1: as before, but integrating (2) between $a_1 \in]a_0, 1[$ and $x \in]a_1, 1[$, and assuming $u'(x) \leq 0$, we obtain that $u'(x) = -|q|\left(\frac{b(a_1)}{b(x)}\right)^{\frac{1}{p-1}}$. Integrating between $x \in]a_1, 1[$ and 1, we can defined

$$u(x) = \int_x^1 |q|\left(\frac{b(a_1)}{b(x)}\right)^{\frac{1}{p-1}} dx \text{ for any } x \in]a_1, 1[.$$

Since $u\left(a_1\right)=1$, $\frac{1}{|q|}\int_{a_1}^{1}|q|\left(\frac{b(a_1)}{b(x)}\right)^{\frac{1}{p-1}}dx$. Analogously, we define the continuous function $h\left(a\right)=1-|q|\int_{a}^{1}\left(\frac{b(a)}{b(x)}\right)^{\frac{1}{p-1}}dx$ to conclude from Bolzano's Theorem that there exists $a_1\in]a_0,1[$ such that the function

$$u\left(x\right)=\int_{x}^{1}|q|\left(\frac{b\left(a_1\right)}{b\left(x\right)}\right)^{\frac{1}{p-1}}dx \text{ for any } x\in]a_1,1[,$$

verifies the problem (2) restricted to $\Omega_1=]a_1,1[$.

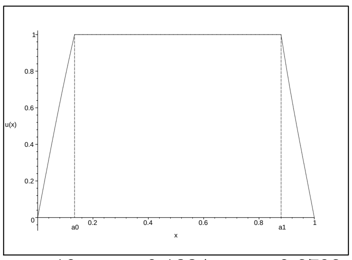
Recalling, we have proved that the function u defined as follow, it is a solution of problem (2):

$$\begin{aligned} u\left(x\right)&=\int_{0}^{x}|q|\left(\frac{b\left(a_0\right)}{b\left(s\right)}\right)^{\frac{1}{p-1}}ds \text{ for any } x\in]0,a_0[,\\ u\left(x\right)&=1 \qquad\qquad\qquad\qquad \text{for any } x\in[a_0,a_1]\\ u\left(x\right)&=\int_{x}^{1}|q|\left(\frac{b\left(a_1\right)}{b\left(s\right)}\right)^{\frac{1}{p-1}}ds \text{ for any } x\in]a_1,1[,\end{aligned}$$

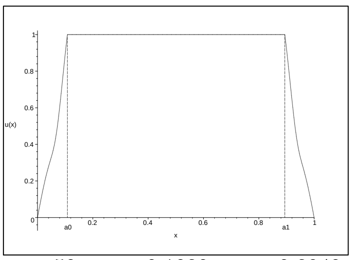
with a_0 and a_1 solutions of nonlinear equations

$$\begin{aligned} 1&=\int_{0}^{a_0}|q|\left(\frac{b\left(a_0\right)}{b\left(s\right)}\right)^{\frac{1}{p-1}}ds\\ 1&=\int_{a_1}^{1}|q|\left(\frac{b\left(a_1\right)}{b\left(s\right)}\right)^{\frac{1}{p-1}}ds\end{aligned}$$

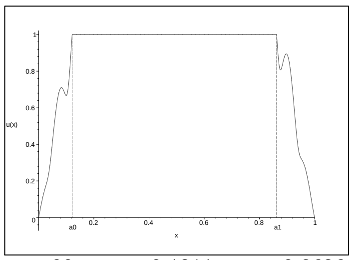
respectively. For the particular case of $b\left(x\right)=2+\cos\left(nx\right),\quad p=4,\quad q=8$ and $n=10, n=50, n=90$, one gives a graphics with the numerical solutions. We can observe, that, there isn't symmetric for the solutions. Similar approach can be used to obtain numerical solution for the radial case.



$n=10, a_0:=0{,}1334, a_1:=0{,}8798$



$n=50, a_0:=0{,}1080, a_1:=0{,}8940$



$n=90,\ a_0:=0{,}1211, a_1:=0{,}8626$

Acknowledgments

Research partially supported by the project MTM2008-06208 of the Ministerio de Ciencia e Innovacion, Spain. The participation in XXI C.D.E.Y.A. / XI C.M.A. was supported by the program *Convocatoria de Ayudas para la Presentación de Ponencias y Comunicaciones en Congresos y otras Reuniones Científico-Técnica* of Universidad Politécnica de Madrid, Spain.

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