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Blow-up in Functional Partial Differential Equations with large amplitude memory terms

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Resumen

In this contribution we use a semigroup approach to analyze blow-up phenomena that appear in elliptic PDE with delay terms on the *boundary* of the domain, that is, with *dynamic boundary conditions*: $\Delta u(t,\mathbf{x}) = 0$, for $(t,\mathbf{x}) \in (0,+\infty)\times\Omega$, $\partial_t u(t,\mathbf{x}) + \partial_n u(t,\mathbf{x}) = b(t)u(t-\tau,\mathbf{x})$, for $(t,\mathbf{x}) \in (0,+\infty)\times\partial\Omega$, $u(\theta,\mathbf{x}) = u_0(\theta,\mathbf{x})$, *initial function*, for $(\theta,\mathbf{x}) \in [-\tau,0]\times\Omega$. The existence of solutions beyond $t = t^*$ in some generalized (L^p) sense is also addressed.

1. Introduction

The purpose of this contribution is to analyze the effect of delay terms in some linear diffusion models with dynamic boundary conditions. More specifically, we consider elliptic equations like

$$\text{(DBC)}\quad \begin{cases} \Delta u(t,\mathbf{x}) = 0 & \text{for } (t,\mathbf{x}) \in (0,+\infty)\times\Omega \\ \partial_t u(t,\mathbf{x}) + \partial_n u(t,\mathbf{x}) = b(t)u(t-\tau,\mathbf{x}) & \text{for } (t,\mathbf{x}) \in (0,+\infty)\times\partial\Omega \\ u(\theta,\mathbf{x}) = u_0(\theta,\mathbf{x}) \quad \text{initial function} & \text{for } (\theta,\mathbf{x}) \in [-\tau,0]\times\Omega \end{cases} \qquad (1)$$

on a smooth bounded domain $\Omega \subset \mathbb{R}^n$, where $\partial_n u$ denotes the outer normal derivative of u at the boundary $\partial\Omega$. The term "dynamic boundary condition" refers to the presence of the time derivative of the unknown u in the boundary condition, and they arise when the values of u on $\partial\Omega$ "drive" the solution on the interior of the domain via some stationary (elliptic) or dynamic (parabolic) method. From our viewpoint, the important fact is that both elliptic and parabolic problems with dynamic boundary conditions can be stated in terms of *analytic semigroups,* a fact which enables a unified treatment of both and provides an essential tool: the *variation of constants formula* for nonhomogeneous problems, with the possibility of defining "mild" and other types of generalized solutions.

The function b appearing in the right-hand side of the dynamic boundary condition is supposed to be non-integrable and blow up at some time $t^* > 0$ like $1/|t-t^*|^{\alpha+1}$ with $0 < \alpha < 1$. We prove that this strong singularity of b induces an analogous blow-up behavior of the solution, and study the possibility of extending the solution (in some generalized sense) beyond t^*.

In order to study this problem we adopt a very common viewpoint in the theory of differential equations with discontinuous non-integrable right-hand sides, as stated, for instance, in [11], which amounts to expressing the non-integrable terms as higher-order distributional derivatives of locally integrable functions. Specifically, function b is assumed to be the distributional derivative of a function $B : (0,\infty) \to \mathbb{R}^+$ such that:

Hypothesis (H): $B \in L^p_{\text{loc}}(0,\infty)$ for some $1 < p < \infty, B$ is C^1 on $(0,\infty)$ except at a given value $t^* > 0$ and $B(0) = 0$.

Obviously, the third one is just a normalization condition. In order to fix the ideas, we will frequently assume a specific form for B :

Hypothesis (H1) : $B(t) = \dfrac{C}{|t-t^*|^\alpha} + m(t), C > 0, 0 < \alpha < 1$ and $m \in C^1(0,\infty)$.

Our main result is the following:

Theorem 1 *Let $\tau > t^* > 0$ and b satisfy hypothesis (H). Then (DBC) has a unique generalized solution on $(0,\infty)$ for a large class of initial functions u_0. This solution belongs to $L^p_{loc}(0,\infty)$ and blows up at $t = t^*$ like $B(t)$.*

A good part of our study uses functional-analytic methods which work for arbitrary $B \in L^p(0,\tau-\delta)$ for some $0 < \delta < \tau - t^*$ and extended in a C^1 way beyond $\tau - \delta$. This means that the specific form (H1) is used to have a concrete example in mind.

2. Summary of previous results: ODE with delay

Let us consider the ODE with delay

$$\begin{cases} u'(t) = B'(t)u(t-\tau) & \text{for } t \geq 0 \\ u(\theta) = \xi(\theta) & \text{given for } -\tau \leq \theta \leq 0 \end{cases} \tag{2}$$

where $u(t)$ is a vector in $\mathbb{R}^n$ and B satisfies hypothesis (H). Integrating formally on $[0,t]$,

$$u(t) = u(t,\xi) = \xi(0) + \int_0^t B'(s)\xi(s-\tau)ds \quad \text{for } t \in [0,\tau]$$

Hence $u(t,\xi)$ blows up at t^* like $B(t)$.for most initial functions ξ. In order to address the question as to whether u can be **continued beyond** t^*, it is clear that a *weak formulation* is required, as follows:

2.1. Weak formulation: associated neutral functional differential equation

Formally we can write

$$B'(t)u(t-\tau) = [B(t)u(t-\tau)]' - B(t)u'(t-\tau)$$

and then express the original equation as follows

$$\text{(ANE)} \quad \begin{cases} \frac{d}{dt}\left[u(t) - B(t)u(t-\tau)\right] = -B(t)u'(t-\tau), & t > 0 \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0 \end{cases} \tag{3}$$

which we call the "**associated neutral equation".** As before, this formulation allows for a direct integration on the first time interval $[0, \tau]$:

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)\xi'(s-\tau)ds \quad \text{for } t \in [0, \tau]$$

and observe that for the integral to behave well need ξ' to be defined a.e. and belong to $L^q(-\tau, 0)$, where $1/p + 1/q = 1$. As a consequence, u is in $L^p(0, \tau)$ and $u(t) - B(t)\xi(t-\tau)$ is the indefinite integral of an L^1 function. Therefore may write the following "asymptotic expansion"

$$u(t) = B(t)\xi(t-\tau) + AC$$

where "AC" stands for "absolutely continuous".

Remark 2 *The singularity of u at t^* is* ***weaker*** *than that of the coefficient B'.The solution belongs to L^p_{loc} and can be considered as a continuation beyond t^*, at least in an integral (L^p) sense.*

2.2. Abstract evolutionary equation with delay

Assume now that A is the infinitesimal generator of a C^0-semigroup $\{e^{At}\}_{t\geq 0}$ on a Banach space X, B satisfies hypothesis (H), $\tau > 0$ is a given constant with $0 < t^* < \tau$ and $\xi : [-\tau, 0] \to X$ is an initial function whose smoothness properties will be discussed later. We consider the abstract retarded functional differential equation

$$\begin{cases} u'(t) = Au(t) + B'(t)u(t-\tau), & t > 0 \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0 \end{cases} \tag{4}$$

As before, B' is to be interpreted in the sense of distributions (with values in X), that is:

$$\int_{-\infty}^{\infty} B'(t)\phi(t)dt = -\int_{-\infty}^{\infty} B(t)\phi'(t)dt$$

for every $\phi \in C^\infty(\mathbb{R})$ with compact support.

The full machinery of functional differential equations in Banach spaces as developed in [12],[22] and others is not needed in this case since the "method of stepsçan be directly applied: for $t \in [0, \tau]$, the nonhomogeneous Cauchy problem

$$u'(t) = Au(t) + B'(t)\xi(t-\tau) \quad \text{for } t \in [0, \tau]$$

can be solved under the appropriate conditions. Once this solution on $[0, \tau]$ is obtained, we proceed similarly on $[\tau, 2\tau]$

$$u'(t) = Au(t) + B'(t)u(t-\tau) \quad \text{for } t \in [\tau, 2\tau]$$

and so on.

2.3. Generalized mild solutions

The variation of constants formula on the first interval $[0, \tau]$ gives the explicit expression

$$u(t) = e^{At}\xi(0) + \int_0^t e^{A(t-s)} B'(s)\xi(s-\tau)ds,\ t \in [0, \tau]$$

By formally integrating by parts,

$$\int_0^t e^{A(t-s)} B'(s)\xi(s-\tau)ds = e^{A(t-s)} B(s)\xi(s-\tau)\Big|_{s=0}^{s=t} -$$
$$-\int_0^t B(s)\left[-Ae^{A(t-s)}\xi(s-\tau) + e^{A(t-s)}\xi'(s-\tau)\right] ds =$$

$$= B(t)\xi(t-\tau) - \int_0^t B(s)\left[-Ae^{A(t-s)}\xi(s-\tau) + e^{A(t-s)}\xi'(s-\tau)\right] ds.$$

Therefore $u(t) = e^{At}\xi(0) + \int_0^t e^{A(t-s)} B'(s)\xi(s-\tau)ds =$

$$= e^{At}\xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)\left[-Ae^{A(t-s)}\xi(s-\tau) + e^{A(t-s)}\xi'(s-\tau)\right] ds$$

On the other hand, since B is smooth on $[\tau, \infty)$, the standard variation of constants formula (without integration by parts) suffices for $t \geq \tau$. We thus may define:

Definition 3 *A function $u : [0, \infty) \to X$ is a* generalized mild solution *of 4 if it is continuous except at $t = t^*$ and the following integrals are well defined:*

$$u(t) = e^{At}\xi(0) + B(t)\xi(t-\tau) -$$
$$\int_0^t B(s)\left[-Ae^{A(t-s)}\xi(s-\tau) + e^{A(t-s)}\xi'(s-\tau)\right] ds \quad \textit{for } t \in [0, \tau]$$
$$u(t) = e^{A(t-\tau)}w(\tau) + \int_\tau^t e^{A(t-s)} B'(s)u(s-\tau)ds \quad \textit{for } t \geq \tau$$

The key point, then, is to find the appropriate conditions on the initial function ξ that guarantee the existence of the first integral. In the case of aanalytic semigroups, we can use the fractional powers of operator A and write:

$$\int_0^t B(s)Ae^{A(t-s)}\xi(s-\tau)ds = \int_0^t B(s)A^{1-\beta}e^{A(t-s)}A^{\beta}\xi(s-\tau)ds$$

where $0 < \beta \leq 1$ assuming that $A^{\beta}\xi(t)$ exists, that is, that $\xi(t)$ belongs to X^{β}, the domain of the fractional power A^{β}. Then, since

$$\left\|A^{1-\beta}e^{At}\right\| \simeq \frac{C}{t^{1-\beta}} \quad \text{as } t \to 0$$

the integral becomes meaningful as long as the product $B(t)A^{\beta}\xi(t)$ has the right integrability properties. A choice of these is stated below:

Theorem 4 *Let $1 < p < \infty$, $\tau > t^* > 0$ be given. Assume:*

- *$B : [0, \infty) \to \mathbb{R}$ is a C^1 function on $[0, \infty)\backslash\{t^*\}$ and belongs to $L^p_{loc}(0, \infty)$.*
- *A is the infintesimal generator of an anaytic semigroup $\{e^{At}\}$ on a reflexive Banach space X. The domains of the fractional powers A^{β} are denoted by X^{β}.*

Let $1/q + 1/p = 1,\ 1/r + 1/s = 1/q$ and $\beta > 1/p - 1/s$, and assume that the initial function $\xi : [-\tau, 0] \to X$ belongs to $W^{1,q}(-\tau, 0; X) \cap L^s(-\tau, 0; X^{\beta})$. Then the problem

$$\begin{cases} u'(t) = Au(t) + B'(t)u(t-\tau) & \text{for } t > 0 \\ u(\theta) = \xi(\theta) & \text{for } \theta \in [-\tau, 0] \end{cases}$$

has a unique generalized mild solution on $[0, \infty)$, and it has the property that $u(t) - B(t)\xi(t-\tau)$ is absolutely continuous on $[0, \tau]$. As a consequence, if $\xi(t^* - \tau) \neq 0$, $\|u(t)\|$ blows up at t^* like $B(t)$.

Proof. On the interval $[0, \tau]$ the generalized mild solution is given by

$$\begin{aligned} u(t) &= e^{At}u(0) + B(t)\xi(t-\tau) - \\ &- \int_0^t B(s)\left[-A^{1-\beta}e^{A(t-s)}A^{\beta}\xi(s-\tau) + e^{A(t-s)}\xi'(s-\tau)\right] ds. \end{aligned}$$

By our assumptions on ξ, the function $\sigma \mapsto A^{\beta}\xi(\sigma - \tau)$ belongs to $L^s(-\tau, 0; X)$. On the other hand, since $\left\|A^{1-\beta}e^{A(t-s)}\right\| \leq C(t-s)^{\beta-1}$ on $(0, t)$ for some constant C, we see that $\left\|A^{1-\beta}e^{A(t-\sigma)}\right\|$, as a function of σ, belongs to $L^r(0, t)$ if and only if $1-\beta < 1/r$. Under these hypothese, the convolution

$$\int_0^t A^{1-\beta}e^{A(t-s)}A^{\beta}\xi(s-\tau)ds$$

belongs to $L^z(0, t)$, where $1/z = 1/r + 1/s$. But $1/r + 1/s = 1 - 1/p = 1/q$, so $z = q$, the conjugate exponent of p. Therefore, the product of $B(t)$ by the above convolution is an integrable function of t and the first term of the integral is well defined and defines an absolutely continuous function from $[0, \tau]$ to X if X is reflexive (see, e.g., [19]).

The second term of the integral is easier to consider, since $\sigma \mapsto \xi'(\sigma - \tau)$ belongs to $L^q(-\tau, 0; X)$ by assumption.

We have thus proved that u is well defined on $[0, \tau]$ and has the form

$$u(t) = B(t)\xi(t - \tau) + \text{ absolutely continuous}$$

an is therefore integrable on $[0, \tau]$ (recall that the Sobolev embedding theorems imlpy that ξ is continuous on $[0, \tau]$). For $t \in [\tau, 2\tau]$, the standard variation-of.constants representation gives no problem, since B' is continuous for $t > t^*$ and $u(t - \tau)$ is integrable on $[\tau, 2\tau]$, as shown before. It is now clear that for $t \geq 2\tau$ the argument can be applied without further modification. This finishes the proof of the theorem.

Remark 5 *It may be difficult to characterize the fractional powers of the operators involved, as well as the L^q, L^s integrability contidions of the theorem. On the other hand, very precise results of this type are needed mostly for* ***nonlinear*** *problems, where more or less sophisticated fixed point arguments are regularly used. But our problem is linear, and for most applications a much simpler version will be enough. That is what we will do when stating the results for models with dynamic boundary conditions in the next section.*

3. Elliptic problems with dynamic boundary conditions

In this section we give a short summary of the analytic semigroups appearing in elliptic problems with dynamic boundary conditions. The parabolic case will be treated elsewhere.

The key ingredient is the fact that our initial equation (EDBC) may be reduced to the most basic case

$$\begin{cases} \Delta u(t, \mathbf{x}) = 0 & \text{for } (t, \mathbf{x}) \in (0, +\infty) \times \Omega \\ \partial_t u(t, \mathbf{x}) + \partial_n u(t, \mathbf{x}) = 0 & \text{for } (t, \mathbf{x}) \in (0, +\infty) \times \partial\Omega \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) \text{ given} & \text{for } \mathbf{x} \in \partial\Omega \end{cases}$$

by using semigroup theory. According to [1], [7], [9] and others, a semigroup can be associated to this problem in a number of ways, some of them using very complicated function spaces of Besov Type. For our purposes we will stick to the simplest formulation found so far, which is given in [3].

Given the bounded smooth domain Ω, let us denote $\mathcal{K} : L^2(\partial\Omega) \to L^2(\Omega)$ the *harmonic extension* of a function $v \in L^2(\partial\Omega)$. By definition, $\mathcal{K}v = u$ is the solution of the elliptic non-homogeneous problem

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u = v & \text{in } \partial\Omega \end{cases}$$

Operator $\mathcal{K}$, sometimes called the "capacity operator", has many nice properties. The only one we need here is the fact that the normal derivative $\partial_n(\mathcal{K}v)$ of the harmonic extension is well defined when the boundary data v belongs to $H^{1/2}(\partial\Omega)$ ([3]).

Let H denote the Hilbert space $L^2(\partial\Omega)$. Then we have the following fundamental result (see [3], Lemma 4.1):

Theorem 6 *Let A be the following linear operator on $H = L^2(\partial\Omega)$:*

$$Av = \frac{\partial}{\partial n}(\mathcal{K}v) \quad \textit{with domain} \quad D(A) = \{v \in H^{1/2}(\partial\Omega) : \frac{\partial}{\partial n}(\mathcal{K}v)|_{\partial\Omega} \in L^2(\partial\Omega\}$$

Then $-A$ is selfadjoint and generates an analytic semigroup on $L^2(\partial\Omega)$.

The relation of this semigroup and equation (EDBC) is the following: Let $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ denote the *trace operator.* Let $v_0 = \gamma(u_0) \in H$, and assume that (EDBC) has a solution $u(t)$. Then $v(t) = e^{At}v_0$ is the trace of $u(t)$, that is, $v(t) = \gamma(u(t))$.In other words, the solution $u(t, .)$ of the nonhomogeneous problem

$$(NH) \quad \begin{cases} \Delta u(t, \mathbf{x}) = 0 & \text{for } (t, \mathbf{x}) \in (0, +\infty) \times \Omega \\ \partial_t u(t, \mathbf{x}) + \partial_n u(t, \mathbf{x}) = f(t, \mathbf{x}) & \text{for } (t, \mathbf{x}) \in (0, +\infty) \times \partial\Omega \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) & \text{for } \mathbf{x} \in \partial\Omega \end{cases}$$

(where $f : (0, \infty) \to H = L^2(\partial\Omega)$ is an integrable function) can be simply expressed by means of the variation of constants formula: $u(t, .)$ is a solution of (NH) if and only of its trace $v(t) = \gamma(u(t))$ satisfies

$$v(t) = e^{At}v_0 + \int_0^t e^{A(t-s)}f(s)ds$$

Hence the semigroup techniques of Section 2 can be applied to the elliptic dynamic boundary value problem with delay

$$\text{(EDBC)} \quad \begin{cases} \Delta u(t, \mathbf{x}) = 0 & \text{for } (t, \mathbf{x}) \in (0, +\infty) \times \Omega \\ \partial_t u(t, \mathbf{x}) + \partial_n u(t, \mathbf{x}) = b(t)u(t - \tau, \mathbf{x}) & \text{for } (t, \mathbf{x}) \in (0, +\infty) \times \partial\Omega \\ u(\theta, \mathbf{x}) = u_0(\theta, \mathbf{x}) \quad \text{initial function} & \text{for } (\theta, \mathbf{x}) \in [-\tau, 0] \times \partial\Omega \end{cases}$$

where $b = B'$ in the sense of distributions and B satisfies hypothesis (H). The semigroup formulation for $v = \gamma(u)$ is

$$\begin{cases} v' = Av + B'(t)v(t - \tau), & t > 0 \\ v(\theta) = \xi(\theta) = \gamma(u_0), & -\tau \leq \theta \leq 0 \end{cases}$$

For ξ in the appropriate function space $W^{1,q}(-t, 0; X) \cap L^s(-\tau, 0; X^\beta)$ the general theorem can be applied, $v(t) - B(t)\xi(t - \tau)$ is continuous on $[0, \tau]$ and there is blow-up at t^* in the $L^2(\partial\Omega)$ norm if $u_0(t^* - \tau, .)$ does not vanish on a set of positive measure of $\partial\Omega$.

In order to avoid technicalities on the function spaces involved, including fractional powers, we will just state a simple version of the above result:

Proposition 7 *Let $\xi : [-\tau, 0] \times \partial\Omega \to \partial\Omega$ be C^1 and let $S = \{\mathbf{x} \in \partial\Omega : \xi(t^* - \tau, \mathbf{x}) \neq 0\}$ be nonempty. Then $|u(t, \mathbf{x})| \to \infty$ as $t \to t^*$ for every $\mathbf{x} \in S$, while $u(t^*, \mathbf{x})$ remains finite if $\mathbf{x} \notin S_0$. Thus, there is* ***simultaneous blow-up*** *on the open and nonempty part of the boundary where $\xi(t^* - \tau, \mathbf{x}) \neq 0$.*

Proof: As a function of t, the map $t \mapsto \xi(t, .)$ belongs to $C^1([-\tau, 0]; C^1(\partial\Omega))$. Since $C^1(\partial\Omega) \subset H^{1/2}(\partial\Omega)$ and the harmonic extension $\mathcal{K}v$ of a $C^1(\partial\Omega)$ function v is in $C^2(\bar{\Omega})$, its normal derivative $\partial_n(\mathcal{K}v)$ is in $C^1(\partial\Omega)$. This means that $C^1(\partial\Omega)$ is contained in $D(A)$, which is contained in any fractional power X^β for $\beta < 1$. Hence ξ is in $W^{1,q}(-\tau, 0; X) \cap L^s(-\tau, 0; X^\beta)$ and the hypotheses of the theorem apply.

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