Min-Max theory

Summer school course by Dan Ketover
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Abstract

Our main goal in this set of notes is to develop the fundamentals of the Simon-Smith min-max theory, whose main result is to prove the existence of minimal, embedded 2-spheres in 3-spheres. Moreover, we will describe some remarkable results which appear as consequences of this theory.

1 Introduction

The classical approach for constructing a minimal surface inside a Riemannian manifold with non-trivial topology involves minimizing the area in a homology class. However, when the manifold is simply connected, this is no longer possible. Thus one seeks to construct a minimal surface by using what is called a min-max procedure. Birkhoff was the first to introduce this method in order to prove the existence of closed geodesics in the 2-sphere $\mathbb{S}^2$ with any Riemannian metric. In the case of surfaces, Almgren and Pitts developed further the min-max theory and were able to prove the existence of embedded minimal surfaces inside a 3-manifold.

Theorem 1.1 (Almgren–Pitts). *Every Riemannian 3-manifold (M^3, g) admits an embedded minimal surface.*

Shortly after, the techniques developed by Almgren–Pitts were refined by Simon-Smith, allowing a control on the topology of the minimal surface obtained. In particular, they proved the following result.

Theorem 1.2 (Simon–Smith [Smi82]). *Every Riemannian 3-sphere admits an embedded minimal 2-sphere.*

The first part of these notes is devoted to the development of the min-max procedure in order to provide a proof of the Simon–Smith theorem.

In the second part we cover other remarkable, more recent, applications of the min-max procedure. First, we will prove the following theorem of Ketover–Marques–Neves on the existence of minimal surfaces realizing the Heegaard genus for the particular case of the real projective space $\mathbb{RP}^3$.

Theorem 1.3 (Ketover–Marques–Neves [KMN20])**.** *Every compact Riemannian 3-manifold of positive Ricci curvature admits an index 1 minimal surface realizing its Heegaard genus.*

An important step in the proof of Ketover–Marques–Neves consists in ruling out certain outcomes of the min-max procedure by means of an area estimate for catenoids.

Then we will adapt the min-max technique in order to detect additional critical points of the area functional. The result of Simon–Smith implies the existence of a minimal 2-sphere in any Riemannian 3-sphere. In his problem section, Yau conjectured that any Riemannian 3-sphere contains at least four different embedded minimal spheres. We will show the following partial result by Halshofer–Ketover.

Theorem 1.4 (Haslhofer–Ketover [HK19])**.** *Let g be a bumpy Riemannian metric on $\mathbb{S}^3$. Then $(\mathbb{S}^3, g)$ contains at least two minimal 2-spheres.*

Finally, we will address the generalised Smale conjecture, which was proven by many different authors for particular cases over the years, and finally solved in full generality by Bamler–Kleiner using Ricci flow.

Theorem 1.5 (Generalised Smale conjecture, Bamler–Kleiner [BK23])**.** *Let (M^3, g) be a closed orientable Riemannian 3-manifold of constant sectional curvature ± 1. Then the inclusion of the isometry group $Isom(M, g)$ into its group of diffeomorphisms $Diff(M)$ is a homotopy equivalence.*

Recently, Ketover–Liokumovich [KL25] gave a different proof of the generalised Smale conjecture using min-max techniques. We will sketch the idea of Ketover–Liokumovich's proof for the case of the real projective space $\mathbb{RP}^3$, which anecdotally was historically the last case to be solved.

Lecture 1

2 Simon–Smith min-max theory

2.1 Minimal surfaces

Let $f : \Sigma^k \to M^n$ be a smooth immersion on a Riemannian manifold (M, g). The intrinsic geometry of Σ is determined by the pullback metric $g_\Sigma = f^*g$ while the extrinsic geometry is described in terms of the second fundamental form $B : \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \to \mathcal{X}^\perp(\Sigma)$ given by

$$B(X, Y) = \left(\overline{\nabla}_X Y\right)^\perp;\ X, Y \in \mathcal{X}(\Sigma)$$

The mean curvature vector H_Σ of the immersion is defined as the trace of the second fundamental form

$$H_\Sigma(p) = \sum_{i=1}^{k} B_p(E_i, E_i) \in T_p\Sigma^\perp$$

where $\{E_i\}$ is an orthonormal basis of $T_p\Sigma$.

Definition 2.1. We say that the smooth immersion $f : \Sigma^k \to M^n$ is **minimal** if $H_\Sigma \equiv 0$.

We will now provide a variational characterization of minimal surfaces. In order to do so, we introduce the following definition:

Definition 2.2. *A compactly supported variation of the immersion* $f : \Sigma \to M$ is a smooth map $F : \Sigma \times (-\epsilon, \epsilon) \to M$ such that

- For every $t \in (-\epsilon, \epsilon)$ the map $F_t : p \in \Sigma \to F(p, t) \in M$ is an immersion.
- $F_0 = f$.
- There exists a compact set $K \subset \Sigma$ such that $F_t(p) = f(p)$ for all $p \in \Sigma \setminus K$ and for all $t \in (-\epsilon, \epsilon)$.

We write $\Sigma_t = F_t(\Sigma)$ and call $X = \partial F_t|_{t=0} \in \Gamma(f^*TM)$ *the variational vector field.*

Remark 2.3. Assume that Σ is two-sided and let $\varphi \in C_c^\infty(\Sigma)$ be a smooth function with compact support. One way to produce compactly supported variations is through the exponential map $F : \Sigma \times (-\epsilon, \epsilon) \to M$ given by $F(p, t) = \exp_p(t\varphi(p)N_\Sigma(p))$, where N_Σ is the normal vector to Σ.

An alternative method to produce variations is by means of the flux Φ_t^X of a compactly supported ambient vector field $X \in \mathcal{X}_c(M)$. In this case, we simply set $F_t(p) = \Phi_t^X \circ f(p)$.

It is of particular interest to study the area functional associated to compactly supported variations.

Theorem 2.4 (The first variation formula)**.** *Let $f : \Sigma \to M$ be a smooth immersion into a Riemannian manifold (M, g). For a compactly supported variation $F : \Sigma \times (-\epsilon, \epsilon) \to M$ we have that*

$$\frac{d}{dt}\mathrm{Area}(\Sigma_t) = \int_{\Sigma_t} div_{\Sigma_t}(\partial_t F)dA_t = -\int_{\Sigma_t} g(H_{\Sigma_t}, \partial_t F)dA_t + \int_{\partial\Sigma_t} g(\partial_t F, \nu_{\partial\Sigma_t})dL_t \tag{1}$$

where H_{Σ_t} is the mean curvature of the surface Σ_t and $\nu_{\partial\Sigma_t}$ is the outer conormal vector field of the boundary.

This theorem allows us to understand a minimal surface Σ as a critical point of the area functional under compactly supported variations away from the boundary $\partial\Sigma$, because in that case the first variation formula is

$$\left.\frac{d}{dt}\right|_{t=0}\mathrm{Area}(\Sigma_t) = -\int_\Sigma g(H_\Sigma, X)\, dA. \tag{2}$$

Now we consider the second variational formula in a very particular case, although it can be stated in a more general way.

Theorem 2.5 (The second variation formula)**.** *Let Σ^{n-1} be a closed two-sided minimal hypersurface in an ambient Riemannian manifold (M^n, g). Then, under a variation with normal vector field $X = \varphi N_\Sigma$ we have*

$$\left.\frac{d^2}{dt^2}\right|_{t=0}\mathrm{Area}(\Sigma_t) = -\int_\Sigma \varphi\left(\Delta_\Sigma\varphi + |A|^2\varphi + Ric^M(N_\Sigma, N_\Sigma)\varphi\right)\, dA, \tag{3}$$

From the second variation formula we recognize the Jacobi operator defined on the space of smooth functions $C^\infty(\Sigma)$

$$\mathcal{L}_\Sigma := \Delta_\Sigma + |A|^2 + \mathrm{Ric}^M(N_\Sigma, N_\Sigma). \tag{4}$$

An eigenfunction $\varphi \in$ with eigenvalue $\lambda \in \mathbb{R}$ is a non-trivial function satisfying $\mathcal{L}_\Sigma\varphi + \lambda\varphi = 0$. From spectral theory [CM11], the Jacobi operator $\mathcal{L}_\Sigma$ is self-adjoint with discrete spectrum

$$\lambda_1 \leq \lambda_2 \leq \cdots \to \infty.$$

In particular, the dimension of each eigenspace E_λ is finite dimensional.

Definition 2.6 (Stability)**.** A minimal hypersurface $\Sigma^{n-1} \subset M^n$ is called *stable*, if all eigenvalues λ_i of the Jacobi operator are non-negative. Otherwise, if some of the eigenvalues are negative, then we say the hypersurface is *unstable*.

Intuitively, a stable minimal hypersurface is one for which *no variation* decreases its area.

Definition 2.7 (Index)**.** The *index* of an unstable minimal hypersurface Σ is defined as

$$\mathrm{index}(\Sigma) = \sum_{\lambda<0} \dim E_\lambda$$

The index of a minimal hypersurface corresponds to the number of directions which decrease its area in a local direction.

Example 2.8 (The dumbbell). Consider the following manifold that is diffeomorphic to the 3-sphere. This has at least three minimal hypersurfaces, corresponding to: (a) the generator of the pinching class of the dumbbell neck; (b), (c) the equatorial 2-spheres along the balls of the dumbbell; see Figure 1.

The hypersurface (a) is locally area-minimizing, hence stable; the hypersurfaces of types (b),(c) are unstable.

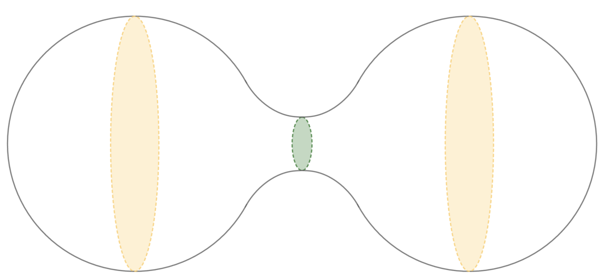

Figure 1: Stable and unstable surfaces in the dumbbell.

Example 2.9. An instructive example is to consider an equatorial sphere $\Sigma = \mathbb{S}^{n-1} \subset \mathbb{S}^n$. This is a minimal submanifold, since it is totally geodesic ($|A| \equiv 0$). Moreover, since we are in an ambient space of sectional curvature 1 we have that

$$\mathrm{Ric}^{\mathbb{S}^n}(N_\Sigma, N_\Sigma) = n - 1,$$

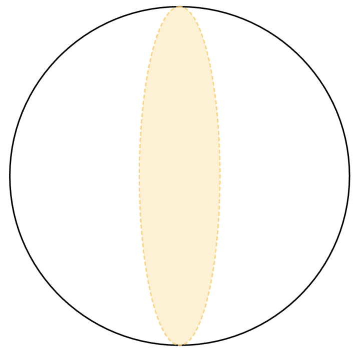

Figure 2: The equatorial sphere $\Sigma = \mathbb{S}^{n-1} \subset \mathbb{S}^n$ is unstable.

and therefore the Jacobi operator is given by

$$\mathcal{L}_\Sigma = \Delta_\Sigma + (n-1).$$

This means that $\phi \in C_0^\infty(\Sigma)$ is an eigenfunction for the Jacobi operator $\mathcal{L}_\Sigma$ with eigenvalue λ if and only if ϕ is an eigenvalue for the laplacian operator Δ_Σ with eigenvalue $\lambda + (n-1)$. Therefore the spectrum of the Jacobi operator shifts to the left with respect to that of the laplacian. Since $\lambda = 0$ is an eigenvalue for the laplacian with associated one dimensional eigenspace, and since the first non-zero eigenvalue of the laplacian is exactly $n-1$ it follows that $\Sigma = \mathbb{S}^{n-1} \subset \mathbb{S}^n$ is a minimal hypersurface with index exactly equal to 1.

More generally, from the second variation formula it can be seen that there are no two-sided Σ^{n-1} closed stable minimal hypersurfaces in a Riemannian manifold with positive Ricci curvature. However, using min-max techniques [IMN18], it is possible to show that for a *generic* metric on the sphere, there exist infinitely many minimal hypersurfaces.

Example 2.10 (The Clifford torus). Consider the 3-sphere with the canonical induced metric

$$\mathbb{S}^3 := \left\{ (z,w) : |z|^2 + |w|^2 = 1 \right\} \subset \mathbb{R}^4 = \mathbb{C}^2$$

we define the *Clifford torus* as

$$\mathbb{S}^{1,1}_{\frac{1}{\sqrt{2}}} = \left\{ (z,w) : |z|^2 = |w|^2 = \frac{1}{2} \right\} \subset \mathbb{S}^3.$$

By a theorem of Urbano [Urb90], this minimal hypersurface has $\mathsf{index}(\mathbb{S}^{1,1}_{\frac{1}{\sqrt{2}}}) = 5$.

2.2 Curvature estimates

During this section we will present some results regarding stability and how this concepts relates to the existence of *limits* of sequences of surfaces.

Theorem 2.11 (Curvature estimate for stable minimal surfaces; Schoen)**.** *Consider an open set $U \subset\subset M^3$. Then, there exists a constant $C_U > 0$ such that for any stable minimal hypersurface $\Sigma \subset U$ with $\partial\Sigma \subset \partial U$, we have*

$$|A|^2(x) \leq C_U \operatorname{dist}(x, \partial U)^{-2}, \qquad \forall\, x \in \Sigma. \tag{5}$$

This result is crucial, because it implies [KD]:

Corollary 2.12. *If $\{\Sigma_i\}$ is a sequence of stable minimal surfaces in U with Area$(\Sigma_i) \leq C$, then there exists a subsequence that converges graphically and smoothly in compact subsets to a stable minimal surface Σ_∞ (possibly with multiplicity). Moreover if each minimal surface Σ_n is embedded then the limit minimal surface Σ_∞ is embedded.*

2.3 Relationship between index and stability

We examine the relationship between index k-minimal surfaces and stable minimal surfaces. A useful observation is the following:

Lemma 2.13. *Let Σ be an index k minimal surface. Let $\{B_i\}_{i=1}^{k+1}$ be a collection of $k+1$ disjoint open sets intersecting Σ. Then, there exists $1 \leq i \leq k+1$ such that Σ is stable in B_i.*

Proof. Suppose by contradiction that for each $i = 1, \ldots, k+1$, there are eigenfunctions φ_i for the Jacobi operator $\mathcal{L}_\Sigma$, such that

$$\mathcal{L}_\Sigma \varphi_i = -\lambda_i \varphi_i, \qquad \operatorname{supp} \varphi_i \subset B_i, \qquad \varphi_i|_{\partial B_i} = 0,$$

where all the λ_i are negative. The quadratic form $Q(\varphi, \psi) := -\int_\Sigma \varphi \mathcal{L}_\Sigma \psi$ defined on the $(k+1)-$dimensional vector space $V := \operatorname{span}\{\varphi_1, \ldots, \varphi_{k+1}\}$ is strictly negative-definite, i.e., $Q|_V$ is strictly negative-definite. Indeed we have that on one hand

$$Q(\varphi_i, \varphi_i) = \lambda_i \int_\Sigma \varphi_i^2 < 0 \quad \text{for any } i,$$

while on the other hand

$$Q(\varphi_i, \varphi_j) = \int_\Sigma \varphi_i \mathcal{L} \varphi_j = \lambda_j \int_\Sigma \varphi_i \varphi_j = 0,$$

because for $i \neq j$ the corresponding eigenfunctions have disjoint supports, so $\varphi_i \varphi_j \equiv 0$. But then, this is equivalent to saying that index$(\Sigma) \geq k+1$, a contradiction. Therefore it must be that the minimal surface Σ is stable in at least one of the sets B_i. ☐

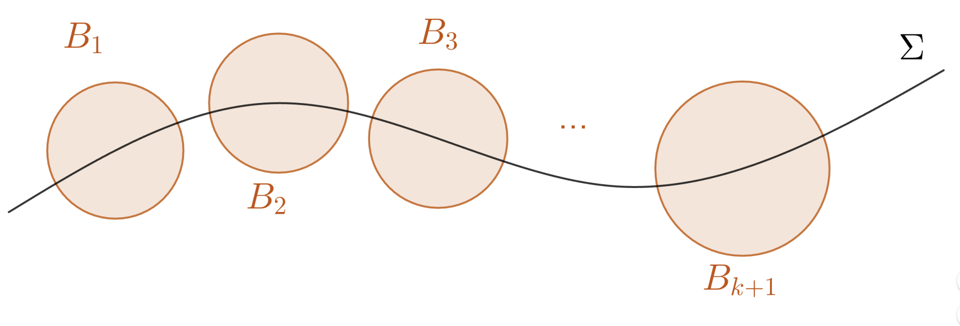

Figure 3: Σ must be stable on at least one of the open sets B_i.

From this, we deduce:

Theorem 2.14 (Dichotomy property)**.** *Let $\{\Sigma_i\}$ be a sequence of index 1 minimal surfaces inside a compact manifold M. Then, if we denote by $B_r(p)$ the geodesic ball of radius r centered at p, up to subsequence, one of the following assertions holds:*

(i) There exists a point $p \in M$ and a sequence of radii $r_j \searrow 0$ such that for each j the sequence $\{\Sigma_i\}$ is stable in $M \setminus B_{2r_j}(p)$ for sufficiently large i.

(ii) There exists $r_0 > 0$ such that the sequence $\{\Sigma_i\}$ is stable in all balls of radius less or equal than r_0.

Proof. Suppose that (i) does not hold. This means that for each point $p \in M$ there exist a radius $r(p) > 0$ such that up to subsequence $\{\Sigma_i\}$ is unstable in $M \setminus B_{2r(p)}$, and therefore by Lemma 2.13 it follows that $\{\Sigma_i\}$ is stable in $B_{r(p)}$. By compactness of the manifold M we can take a finite subcover that fills up the entire ambient manifold,

$$M = \bigcup_{i=1}^{k} B_{r(p_k)}(p_k).$$

We may then choose any r_0 less than the Lebesgue number of the covering and therefore any ball of radius less than r_0 is contained in one of the balls $B_{r(p_k)}(p_k)$ of the cover. Since each of the $\{\Sigma_i\}$ is stable in $B_{r(p_k)}(p_k)$, it is also stable in this ball of radius less than r_0, satisfying property (ii) as desired. □

Remark 2.15. An analogous version of this result holds if the surfaces Σ_i in the above sequence are assumed to have index $\leq k$, for any k; the only modification is that in part (i), one might have to remove more points, meaning that the surfaces are stable away from a finite collection of points.

Example 2.16 (The blow-down of the catenoid). This dichotomy is exemplified by the catenoid $C \subset \mathbb{R}^3$. For a sequence of $\lambda_i \searrow 0$, we can consider the sequence $\Sigma_i := \lambda_i C$ of blow-down rescalings of the catenoid at the origin. We claim that this sequence satisfies the first property (i) of Theorem 2.14. In fact, we can choose the point p to be the origin, and the sequence of radii $r_j = \lambda_j$. Then for each j, outside the ball B_{2r_j} for sufficiently large i the catenoids Σ_i will have two connected components which are all graphical and hence stable. These connected components converge to a *horizontal* plane, which is also stable.

Remark 2.17. The dichotomy property can be used to prove that the doubly periodic Scherk surface $\mathcal{S}$ has unbounded index. Indeed for a sequence of $\lambda_i \searrow 0$, we accordingly form the sequence $\Sigma_i := \lambda_i \mathcal{S} \cap B_1$, where B_1 denotes the closed unit ball. It is interesting to notice that this sequence converges in B_1 to the intersection of two transversal disks which is not embedded along a *vertical* segment Γ. If we suppose by contradiction that $\mathcal{S}$ has bounded index, then either (i) or (ii) should hold. In any case we can find an open set U such that $U \cap \Gamma \neq \varnothing$ so that $\Sigma_i|_U$ should be stable. This sequence of stable, embedded minimal surfaces have to converge in U to a pair of intersecting disks, which is impossible by Corollary 2.12.

Figure 4: The blowdown of the catenoid.

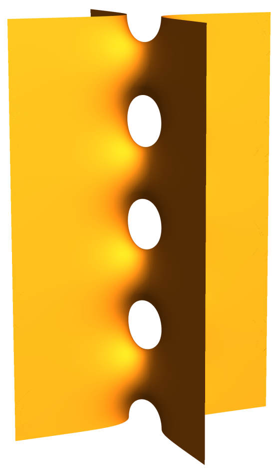

Figure 5: The doubly periodic Scherk surface has unbounded index.

2.4 The min-max construction

Our goal now is to introduce the notions and tools to provide a proof of the Simon-Smith theorem 1.2. We adopt for the rest of the discussion a special definition of sweepout of a Riemannian 3-sphere (M^3, g) by 2-spheres which will serve our purposes. However, it is important to point out that there is a much more developed definition of sweepout by the so-called generalized family of surfaces (see [CDL03] for more details).

Definition 2.18 (Sweepout by spheres)**.** Let (M^3, g) be a Riemannian 3-sphere. A *sweepout* of M^3 is a smooth map $F(x,t) : \mathbb{S}^3 \subset \mathbb{R}^3 \times \mathbb{R} \to M^3$ such that for every $t \in (-1,1)$ the surface $\Sigma_t \equiv F(\cdot, t)$ is an embedded 2-sphere.

Example 2.19. An example of sweepout on the canonical sphere $(\mathbb{S}^3, g_{\text{round}})$ is just given by the identity map $F : \mathbb{S}^3 \to \mathbb{S}^3$. The family of embedded spheres $\Sigma_t = F(\cdot, t)$ is

$$\Sigma_t := \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{S}^3 : x_4 = t \right\}.$$

We also have a general procedure for producing sweepouts, by using isotopies. Concretely, let us consider a smooth map

$$\Psi : M \times [-1, 1] \to M$$

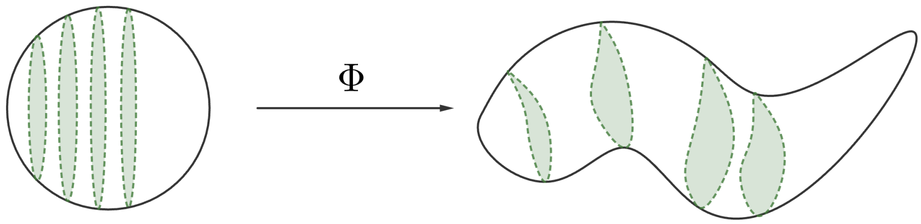

Figure 6: Illustration of a sweepout in (M^3, g).

such that for each t, the restriction $\Psi(\cdot, t) : M \to M \in \mathrm{Diff}_0(M)$ is a diffeomorphism of M in the connected component of the identity map. It is immediate to see that if $\{\Sigma_t\}_t$ is a sweepout, then so is the family $\{\Psi(t, \Sigma_t)\}_t$. We then define

$$\Pi = \{\text{smallest collection of sweepouts which is closed under isotopies}\}\,.$$

Definition 2.20. The *min-max width* of Π is defined as

$$\omega = \inf_{\Sigma_t \in \Pi} \sup_{t \in [-1,1]} \mathrm{Area}(\Sigma_t). \tag{6}$$

We have the following result:

Theorem 2.21. *Let $F(x, t)$ be a diffeomorphism. Then, the min-max width ω associated to the sweepout $F(x, t)$ is positive.*

Proof. Pick an element $\tilde{\Sigma}_t \in \Pi$ and consider the 3-ball Ω_t that it bounds, so that $\partial\Omega_t = \tilde{\Sigma}_t$. We consider the function $f(t) := \mathrm{Vol}(\Omega_t)$. Since the surfaces $\tilde{\Sigma}_t$ are the result of applying an isotopy to the *original* surfaces $\Sigma_t := F(\cdot, t)$, we must have

$$f(-1) = 0, \qquad f(1) = \mathrm{Vol}(M),$$

where we use that $F(x, t)$ is a diffeomorphism. Hence, by continuity, we can produce some $t_0 \in [-1, 1]$ such that

$$\mathrm{Vol}(\Omega_{t_0}) = \frac{1}{2}\,\mathrm{Vol}(M).$$

By the isoperimetric inequality on the 3-sphere M, we know that there exists a constant $C > 0$ such that any closed surface Σ bounding a region Ω satisfies

$$\mathrm{Vol}(\Omega)^{1/3} \leq C \cdot \mathrm{Area}(\Sigma)^{1/2}$$

where Ω denotes the smaller of the two regions bounded by Σ. Applying this result to the surface $\tilde{\Sigma}_{t_0}$, we are guaranteed that

$$\frac{1}{C}\left(\frac{1}{2}\,\mathrm{Vol}(M)\right)^{1/3} = \frac{1}{C}\,\mathrm{Vol}(\Omega_{t_0})^{1/3} \leq \mathrm{Area}(\tilde{\Sigma}_{t_0})^{1/2}.$$

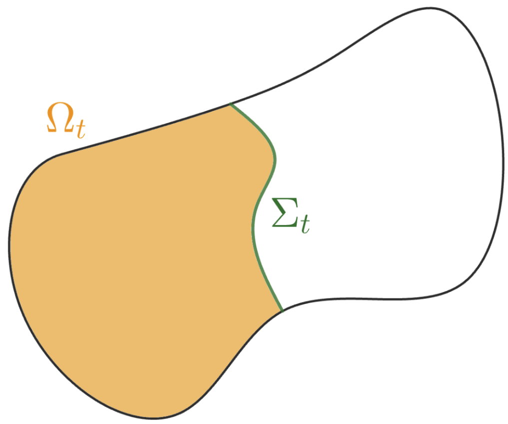

Figure 7: Region Ω_t bounded by the slice Σ_t.

In particular, this shows that we can bound

$$0 < C^{-2}\left(\frac{1}{2}\operatorname{Vol}(M)\right)^{2/3} \leq \operatorname{Area}(\tilde{\Sigma}_{t_0}) \leq \sup_{t\in[-1,1]}(\tilde{\Sigma}_t)$$

where the left-hand side is independent of $\tilde{\Sigma}_t \in \Pi$. Thus

$$0 < C^{-2}\left(\frac{1}{2}\operatorname{Vol}(M)\right)^{2/3} \leq \inf_{\tilde{\Sigma}_t\in\Pi}\sup_{t\in[-1,1]}\operatorname{Area}(\tilde{\Sigma}_t) = \omega$$

□

The geometrical interpretation of the fact that the width is positive for a certain sweepout, is that there will exist a sequence of minimal surfaces of the sweepout converging to a minimal surface whose area realizes the min max width. Therefore the strategy to find the minimal sphere $\mathbb{S}^2$ of Simon-Smith's theorem 1.2 will be to find a sequence of minimal surfaces whose areas are converging to the min-max width, and then achieve the desired minimal sphere $\mathbb{S}^2$ as a limit surface of this sequence. More precisely we have the following two notions

Definition 2.22 (Minimizing sequence)**.** A *minimizing sequence* is a sequence of sweepouts $\{\Sigma_t\}^i \in \Pi$ such that

$$\sup_{t\in[-1,1]}\operatorname{Area}(\{\Sigma_t\}^i) \to \omega$$

Definition 2.23 (Min-max sequence)**.** A *min-max sequence* $\{\Sigma^i_{t_i}\}$ is a sequence of surfaces obtained from a minimizing sequence such that it satisfies

$$\operatorname{Area}(\Sigma^i_{t_i}) \to \omega.$$

Our hope is that some min-max sequence will converge to a minimal surface. One of the challenges is that the sequence $\Sigma^i_{t_i}$ satisfies no PDE so it is hard to control the regularity of the limit objects. The other problem is that not all min-max sequences might capture the limit we are looking for, even though they converge to the desired area. For instance, working with a min-max of $(\mathbb{S}^3, g_{\text{round}})$ by 2-spheres, we would like our sweepout to detect some kind of equatorial sphere Σ_0, which has $|\Sigma_0| = 4\pi$. However, the problem is that our min-max sequence might end up including some very wiggly surfaces whose slices still satisfy $|\Sigma_t| \approx 4\pi$; see Picture 8 below.

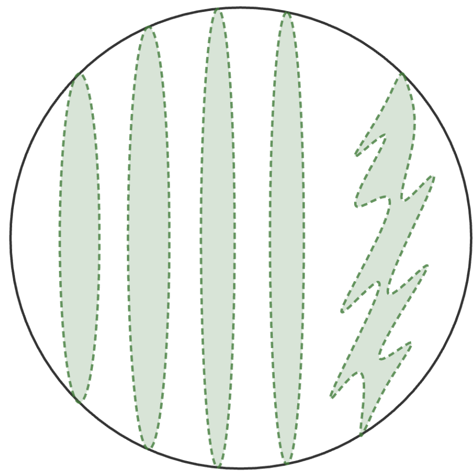

Figure 8: Illustration of a problematic situation of the min-max procedure.

To remedy these problems, we perform the following steps.

The min-max procedure.

(i) **Step 1:** We perform the *pull-tight* of the sequence of sweepouts. This is, we find a minimizing sequence of sweepouts $\{\Sigma_t^j\}$ such that every min-max sequence obtained from it converges to a *stationary varifold*, i.e., a weak measure-theoretic notion of minimal surface that allows for limits which might not be regular, but also enjoys important compactness properties, see Subsection 2.5.

(ii) **Step 2:** We show that, from the pulled-tight sequence of sweepouts, we can extract a min-max sequence $\{\Sigma_{t_j}^j\}$ which is $\frac{1}{j}-$*almost minimizing on annuli*. This is a technical condition that means that the surfaces of the sequence are in some sense close to stable minimal surfaces, see Subsection 2.6.

(iii) **Step 3:** We show that the $\frac{1}{j}-$almost minimizing min-max sequence $\{\Sigma_{t_j}^j\}$ produced in the previous step has regular limits, possibly with multiplicity. More precisely, we show that $\{\Sigma_{t_j}^j\}$ converges in the varifold sense to an integral varifold $\sum_{i=1}^k m_i \mu_{\Gamma_i}^*$, where $\{\Gamma_i\}$ is a collection of embedded pairwise disjoint minimal surfaces, and $m_i \in \mathbb{N}^*$, see Subsection 2.5.

(iv) **Step 4:** To show that the genus of each Γ_i is zero, hence each surface Γ_i is a topological sphere.

2.5 Theory of varifolds

We make an interlude on the definition and properties of varifolds to rigorously treat the steps of the above construction. A good reference for this material is Leon Simon's book on geometric measure theory [LS84]. Given a $k-$dimensional C^1-submanifold of Euclidean space $\Sigma^k \subset \mathbb{R}^N$, we may define a measure μ_Σ on $\mathbb{R}^N$ as

$$\mu_\Sigma(\mathcal{O}) := \mathcal{H}^k(\mathcal{O} \cap \Sigma^k) = \int_{\Sigma\cap\mathcal{O}} 1\, d\mathcal{H}^k, \qquad \mathcal{O} \subset \mathbb{R}^N. \tag{7}$$

where $\mathcal{H}^k$ denotes the $k-$dimensional Hausdorff measure on $\mathbb{R}^N$.

A key insight is that such a surface also defines another measure over the k-**Grassmannian** $\mathbb{G}_k(\mathbb{R}^N) = \mathbb{R}^N \times \mathbb{G}(k, N)$, where $\mathbb{G}(k, N)$ denotes the collection of all $k-$planes on $\mathbb{R}^N$. This measure is defined by

$$\mu^*_\Sigma(\mathcal{O}) := \mathcal{H}^k\left(\{x \in \Sigma : (x, T_x\Sigma) \in \mathcal{O}\}\right), \qquad \mathcal{O} \subset \mathbb{G}_k(\mathbb{R}^N).$$

Here, $T_x\Sigma$ is the tangent space to the surface Σ at the point x.

Definition 2.24 (Varifold)**.** A *k-varifold* is a positive-valued Radon measure on $\mathbb{G}_k(\mathbb{R}^N)$.

Therefore we can naturally see a k-dimensional smooth surface $\Sigma^k \subset \mathbb{R}^N$ as a k-varifold. Notice that we have made the definition of k-varifolds in $\mathbb{R}^N$ instead of varifolds in a general Riemannian manifold M because we can always embed M isometrically into some $\mathbb{R}^N$ for N sufficiently large.

Definition 2.25 (Integral varifold)**.** An *integral k-varifold* is a varifold of the form

$$\sum_{i=1}^{\infty} m_i \mu^*_{\Sigma_i}$$

where $m_i \in \mathbb{N}^*$ and the Σ_i are C^1-submanifolds of $\mathbb{R}^N$.

Definition 2.26 (Rectifiable varifold)**.** A *rectifiable k-varifold* is a countable sum of the form

$$\sum_{i=1}^{\infty} \mu^*_{\Sigma_i, m_i}$$

where each Σ_i is a C^1-submanifold of $\mathbb{R}^N$ and $m_i : \Sigma_i \to \mathbb{R}^+$ is a *multiplicity function.* The Radon measure $\mu^*_{\Sigma_i, m_i}$ is defined as

$$\mu^*_{\Sigma_i, m_i}(\mathcal{O}) := \int_{\{x\in\Sigma_i : (x, T_x\Sigma_i)\in\mathcal{O}\}} m_i(x)\, d\mathcal{H}^k, \qquad \mathcal{O} \subset \mathbb{G}_k(\mathbb{R}^N). \tag{8}$$

When this multiplicity function is $\mathcal{H}^k-$equal to a constant natural number, we have that $\mu^*_{\Sigma_i, m_i} \equiv m_i \mu^*_{\Sigma_i}$.

Varifolds are a convenient generalization of surfaces; they naturally allow for strong compactness theorems. Such results are best stated in the context of the weak-star topology:

Definition 2.27 (Weak-star convergence)**.** We say that *a sequence of varifolds* $\{V_k\}$ *weak-star converges to the varifold* V and denote it by $V_k \overset{*}{\rightharpoonup} V$ if and only if

$$\int f\, dV_k \to \int f\, dV, \qquad \forall\, f \in C_c(\mathbb{G}_k(\mathbb{R}^N)).$$

For a varifold V and a compactly contained subset $\mathcal{U} \subset \mathbb{R}^N$ we define the *mass* of the varifold V on $\mathcal{U}$ as

$$\|V\|(\mathcal{U}) := V(\mathbb{G}_k(\mathcal{U})). \tag{9}$$

where $\mathbb{G}_k(\mathcal{U}) = \mathcal{U} \times \mathbb{G}(k, N)$.

A priori, the notion of convergence of varifolds might be quite pathological:

Example 2.28. Let $\{\Sigma_n\}$ be the sequence of ellipses in $\mathbb{R}^2$ with major axis 1 and minor axis $\frac{1}{n}$. For $n = 1$, we get that Σ_1 is the usual circle $\mathbb{S}^1$, which shrinks as $n \to \infty$. In the limit, we see that $\mu^*_{\Sigma_n} \rightharpoonup 2\mu^*_I$ converges *with multiplicity two* to the varifold μ^*_I associated to the unit interval $[0, 1]$.

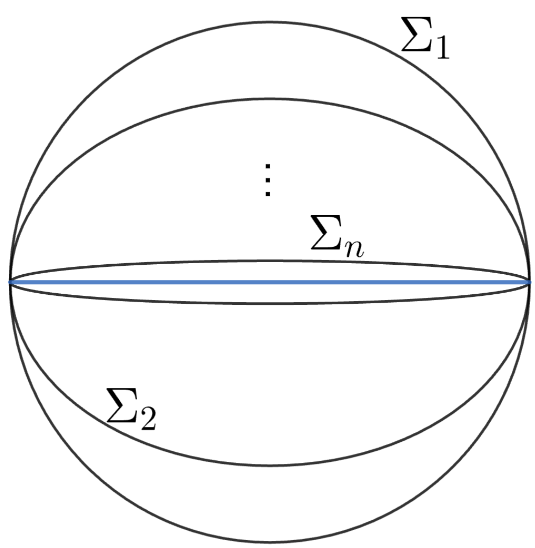

Figure 9: The sequence Σ_n of Example 2.28.

Example 2.29. Consider the sequence of one-dimensional measures μ_j on the plane $\mathbb{R}^2$, given by "sharpening saw-teeth:" they are given by alternating segments at $\frac{\pi}{4}$ angles with height $\frac{1}{j}$ going to 0 as $j \to \infty$. Then, one can show using the Pythagorean theorem that $\mu_j \rightharpoonup \sqrt{2}\mu_I$ in the sense of measures. However, the associated integral varifolds μ^*_j do not converge to a rectifiable varifold: in fact, assume the opposite for the sake of contradiction. Since measure convergence is weaker than varifold convergence, we must have that $\mu^*_j \rightharpoonup \sqrt{2}\mu^*_I$. However this is impossible. First notice that $G_1(\mathbb{R}^2)$ is isomorphic to $\mathbb{R}^2 \times \mathbb{RP}^1$, since $\mathbb{RP}^1$ takes into account the direction of any tangent line on $T\mathbb{R}^2$. Now take the open set $\mathcal{U} := U_0 \times U_1$, where $U_0 \subset \mathbb{R}^2$ is an open set containing the interval $[0, 1]$ and U_1 is an open set of $\mathbb{RP}^1$ containing the *horizontal* line $l_1 = \{y = 0\}$, that is, the line tangent to I. Furthermore, we can take

U_1 so that it does not contain the lines $\{x = y\}$ or $\{x = -y\}$. We define then a function f supported in $\mathcal{U}$ which satisfies that $f|_{I\times\{l_1\}} \equiv 1$. With this in mind, one can check that $\int f d\mu_j^* = 0$, but $\int f\sqrt{2} d\mu_I^* \neq 0$, reaching a contradiction.

Figure 10: The sequence of varifolds μ_j^* of Example 2.29.

We can also consider another example which shows that genus is not preserved in varifold limits:

Example 2.30. Let T_n be a sequence of squares bent as in Figure 11, and assume that they converge to a full torus $\mathbb{T}$ as n increases. Then we can wrap each of the squares T_n with a sphere Σ_n, in a way such that the sequence Σ_n converges to *two copies* of the torus $\mathbb{T}$. In particular, this shows that the genus might grow under varifold convergence.

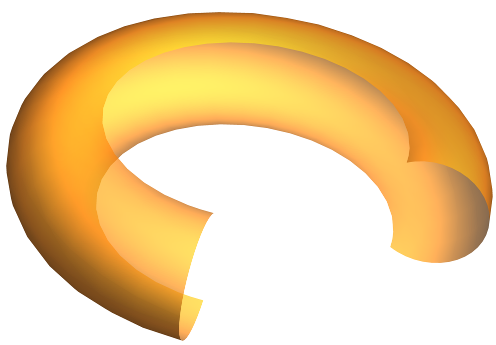

Figure 11: The sequence T_n of Example 2.30

2.5.1 First variation formula for varifolds

Recall that if $\Sigma^k \subset \mathbb{R}^N$ is a smooth submanifold, then any ambient smooth compactly supported vector field $X \in \mathcal{X}(\mathbb{R}^N)$ defines a flow $\{\varphi_t\}$. We thus obtained the first variation of the area under this flow by

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{H}^k(\varphi_t(\Sigma)) = \int_\Sigma \operatorname{div}_\Sigma X \, d\mathcal{H}^k.$$

where the divergence operator is given in terms of the connection $\overline{\nabla}$ in the ambient Euclidean space $\mathbb{R}^{N+1}$

$$\operatorname{div}_\Sigma X(p) = \sum_{i=1}^{k} \left\langle \overline{\nabla}_{E_i} X, E_i \right\rangle; \ \operatorname{span}\{E_1, \cdots, E_k\} = T_p\Sigma$$

In this way, the divergence operator of an ambient vector field $X \in \mathcal{X}(\mathbb{R}^N)$ can be seen as a function on the product space $\mathbb{G}_k(\mathbb{R}^N) = \mathbb{R}^N \times \mathbb{G}(k, N)$

$$\operatorname{div} X : \mathbb{G}_k(\mathbb{R}^N) \to \mathbb{R}, \qquad (x, \pi) \mapsto \operatorname{div}_\pi(x) = \sum_{i=1}^{k} \langle \overline{\nabla}_{E_i} X, E_i \rangle$$

where $\pi = \operatorname{span}\{E_1, \cdots, E_k\} = T_x\Sigma$.

For a $k-$varifold V and X an ambient C^2 vector field, we can prove a *first variation* formula for V (see the full details on [LS84])

$$\delta V(X) = \int \operatorname{div}_\pi X(x)\, dV(x, \pi) \tag{10}$$

Definition 2.31 (Stationary varifold)**.** A varifold V is called *stationary* in the open set $U \subset \mathbb{R}^N$, if it satisfies $\delta V(X) = 0$ for any vector field $X \in C^2_c(U; \mathbb{R}^N)$ that is compactly supported in U.

For an integral varifold $V = \mu^*_\Sigma$, we can prove that the first variation of the area in the direction of an ambient vector field X is given by

$$\delta \mu^*_\Sigma(X) = -\int_\Sigma \vec{H}_\Sigma \cdot X^N + \int_{\partial\Sigma} \langle X, \nu \rangle \tag{11}$$

In principle this formula only makes sense for C^2 surfaces but it can actually be obtained for general varifolds, even if they are not rectifiable (see [LS84] for more details).

Example 2.32 (The triple junction). Consider the *triple junction*, i.e., the 1-varifold in the plane $\mathbb{R}^2$ given by three semi-straight lines S_1, S_2, S_3 with a common origin and making an angle of $\frac{2\pi}{3}$ with one another. Then, the varifold $S := \mu^*_{S_1} + \mu^*_{S_2} + \mu^*_{S_3}$ is stationary. Indeed, the interiors of the lines clearly satisfy $H_{\Sigma_i} = 0$, so they are stationary; hence, any contribution of the first variation must come from the conormal vectors at the boundary of the S_i, which for each of them is just the origin. As a consequence, the first variation of S is given by

$$\begin{aligned} \delta S(X) &= \delta\mu^*_{S_1}(X) + \delta\mu^*_{S_2}(X) + \delta\mu^*_{S_3}(X) \\ &= X(0) \cdot (\vec{\nu}_{S_1} + \vec{\nu}_{S_2} + \vec{\nu}_{S_3}) \\ &= 0 \end{aligned}$$

for any vector field X, because the boundary conormal vectors $\vec{\nu}_{S_i}$ of the S_i satisfy a *force-balancing* condition $\vec{\nu}_{S_1} + \vec{\nu}_{S_2} + \vec{\nu}_{S_3} = 0$. Thus, S is stationary, as desired.

An important reason to study integral varifolds is the key compactness theorem of Allard, which allows one to extract subsequential limits of such objects.

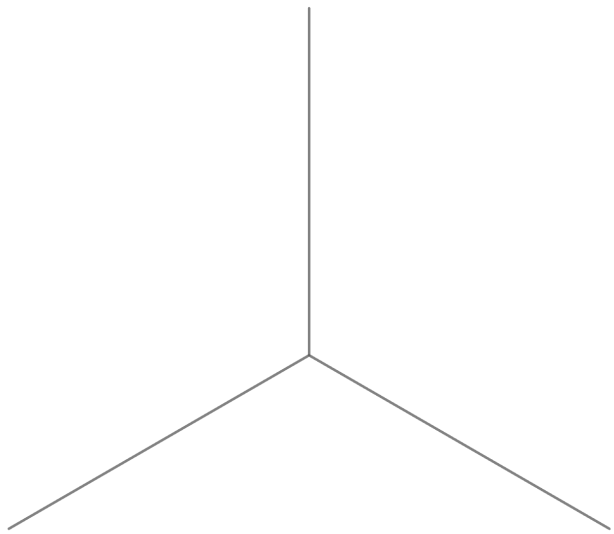

Figure 12: An example of stationary integral varifold: the triple junction.

Theorem 2.33 (Allard's integral compactness theorem [LS84, Theorem 5.8 and Remark 5.9])**.** *Let* $\{\Sigma_i\}$ *be a sequence of stationary integral varifolds in an open set* $U \subset \mathbb{R}^N$*. Suppose that the associated varifold measures have uniformly bounded mass on compact subsets, meaning* $\sup_i \|\Sigma_i\|(W) < +\infty$ *for any* $W \Subset U$*. Then, there exist an integral and stationary varifold* Σ_∞ *such that, up to subsequence,* $\Sigma_i \rightharpoonup \Sigma_\infty$ *in the sense of varifolds.*

Lecture 3

We begin this lecture by providing two examples of sequences of stationary integral varifolds whose limits are also stationary integral, in accordance with Allard's integral compactness theorem.

Example 2.34 (Blowdowns of the catenoid). Let $\lambda_i \searrow 0$ and consider the sequence $\Sigma_i = \lambda_i C \cap B_1(0)$ of blowdowns of the catenoid $C \subset \mathbb{R}^3$ introduced in Example 2.16. In this case, we have that the Σ_i converge to a horizontal disk with multiplicity 2, see Figure 13. This limit is clearly a stationary integral rectifiable varifold in $B_1(0)$.

Figure 13: The blow down of the catenoid.

More generally, we get that the blowdowns converge with multiplicity 2 to the plane $\{y = 0\}$, in the sense of convergence on all compact subsets.

Example 2.35 (Lawson surfaces). For any genus g, let us consider the minimal surfaces in the sphere $\mathbb{S}^3$ of genus g, constructed by Lawson. The Lawson surface of genus g is denoted by $\xi_{g,1}$. These have uniformly bounded mass $\|\xi_{g,1}\| < 8\pi$. As we would expect, the $\xi_{g,1}$ converge, as $g \to \infty$, to the union of two orthogonal equators in $\mathbb{S}^3$ in the sense of varifolds.

2.5.2 The monotonicity formula

Another important property of stationary varifolds is the monotonicity formula:

Proposition 2.36 (Monotonicity formula for varifolds)**.** *Let V be a k-varifold. For any $p \in \mathbb{R}^N$ and $r > 0$ we may define the expression*

$$\Theta(V, p, r) := \frac{\|V\|(B_r(p))}{\omega_n r^n} \tag{12}$$

called the density ratio at p, where ω_n denotes the area of the $n-$dimensional sphere. If the varifold V is ***stationary*** *then the function*

$$r \mapsto \Theta(V, p, r)$$

is ***non-decreasing*** *in r. Thus, in this case, we have a well-defined lower limit*

$$\Theta(V, p) := \lim_{r \downarrow 0} \Theta(V, p, r) \tag{13}$$

called the density of the varifold at p.

Example 2.37 (Examples of density). For example, consider the varifold given by the intersection of two planes. Away from the line of intersection, we have that the intersection is equal to $\Theta(V, p) = 1$; along the line of intersection, the density is given by $\Theta(V, q) = 2$. For another example, the triple-junction varifold considered in Example 2.32 has density 1 at all points along the segments, but density equal to $3/2$ at the point in the origin.

As a consequence of the monotonicity formula, we obtain the following key rectifiability result.

Theorem 2.38 (Rectifiability criterion [LS84, Theorem 8.1.2])**.** *Let V be a stationary varifold such that $\Theta(V, p) \geq c > 0$ for $\|V\|$-a.e. $p \in \mathbb{R}^N$. Then, V is rectifiable and its multiplicity function is given by $\Theta(V, p)$.*

We introduce now the notion of tangent varifold, which is the natural generalization of the tangent space in the case of regular manifolds. Let V be a k-varifold, $p \in \mathbb{R}^N$ be a point and $\lambda > 0$. We will denote by

$$\lambda(V - p) := (\varphi_{p,\lambda})_* V, \tag{14}$$

where $\varphi_{p,\lambda}$ denotes the dilation $y \mapsto \lambda(y - p)$ and $f_* V$ denotes the **pushforward** of V by f, see [CDL03, Section 2.3].

Definition 2.39 (Cone)**.** A k-varifold C is a *cone* if it is invariant under dilation from the origin, that is, if $\lambda C = C$ for any $\lambda > 0$.

Definition 2.40 (Tangent varifold)**.** Let V be a k-varifold and $p \in \mathbb{R}^N$ a point. A *tangent varifold* to V at p is any limit of a sequence of dilations $\lambda_n(V - p)$, where $\lambda_n \nearrow \infty$. We denote the set of tangent varifolds to V at p by $T(p, V)$.

Remark 2.41. Along this course, we will focus on the case of 2-varifolds defined on 3-manifolds. In that case, we slightly modify (14) and we define instead $\varphi_{p,\lambda}$ to be the composition of a dilation and the inverse of the exponential map at p, that is, $\exp_p : \mathbb{R}^3 \mapsto M$, where we identify $\mathbb{R}^3 \equiv T_pM$. Under these conditions, tangent varifolds will be naturally defined on $\mathbb{R}^3$.

Remark 2.42. It is possible to check, using (10), that any tangent varifold to a stationary varifold is also stationary.

Stationary rectifiable varifolds with constant densities are cones:

Lemma 2.43 ([LS84, Remark 5.6.2])**.** *If V is a stationary rectifiable varifold, then V is a cone centered at p if and only if the map $r \mapsto \Theta(V, p, r)$ is constant.*

The following lemma asserts that any tangent varifold to a stationary varifold with a uniformly positive density is a cone.

Lemma 2.44 ([LS84, Corollary 8.5.7])**.** *Let V be a stationary varifold such that $\Theta(V, p) \geq c > 0$ for $\|V\|$-a.e. $p \in \mathbb{R}^N$. Then any tangent varifold to V is a stationary rectifiable cone.*

2.6 Proof of the Simon–Smith min-max theorem

Once we have introduced the notion of varifold and established some of their properties, we can proceed to detail each step in the min-max procedure. We will not include a proof of **Step 1**, but the reader can find one in [CDL03, Proposition 3.1].

Proposition 2.45 (Pull-tight [CDL03])**.** *There exists a minimizing sequence of sweepouts $\{\Sigma_t^j\}$ such that every min-max sequence converges to a stationary varifold.*

Step 4 will not be proven either. This result can be deduced from a general genus bound established by Ketover [Ket19] for the general case of sweepouts by surfaces of arbitrary genus, that we will state properly later in Section 3, see Theorem 3.12. In our case, the genus bound for sweepouts of the 3-sphere by 2-spheres is as follows.

Corollary 2.46 (Genus bound for sweepouts by 2-spheres [Ket19])**.** *Let $\{\Sigma_{t_j}^j\}$ be a $\frac{1}{j}-$almost minimizing min-max sequence of 2-spheres converging to an integral varifold $\sum_{i=1}^k m_i \mu^*_{\Gamma_i}$, where $\{\Gamma_i\}$ is a collection of embedded pairwise disjoint minimal surfaces. Then each minimal Γ_i has genus 0, that is, it is homeomorphic to the 2-sphere.*

This proves Theorem 1.2 once **Steps 2** and **3** are covered.

2.6.1 Proof of Step 3

We will now prove **Step 3** assuming **Step 2**, which will be discussed in the following lecture. Before beginning the proof of **Step 3**, we will introduce the notion of almost minimizing sequence on annuli.

Let $\Psi : M \times [0,1] \to M$ a smooth isotopy, as introduced in Section 2.4. In what follows, we will denote the diffeomorphism corresponding to a certain time $t \in [0,1]$ by $\Psi_t := \Psi(\cdot, t)$.

We will say that the isotopy Ψ is *supported* in a subset $U \subset M$ if the isotopy is constantly the identity away from U, that is,

$$\Psi_t|_{M \setminus U} = \mathsf{id}_{M \setminus U},$$

for all $t \in [0,1]$.

We start by introducing the notion of almost minimizing surface. Intuitively, a surface Σ is almost minimizing if any deformation $\Sigma_t = \Psi_t(\Sigma)$ of Σ such that $\Psi_1(\Sigma)$ has *small area* necessarily contains a slice Σ_{t_0} with $t_0 \in [0,1]$ of *large area*.

Definition 2.47 (Almost minimizing surface)**.** Let $\varepsilon > 0$. A surface Σ in M is ε*-almost minimizing* in an open set $U \subset M$ if there exists no isotopy Ψ_t supported in U, with $\Psi_0 = \mathrm{id}$ and such that

(i) $\|\Psi_t(\Sigma)\| \leq \|\Sigma\| + \frac{\varepsilon}{8}$, and

(ii) $\|\Psi_1(\Sigma)\| \leq \|\Sigma\| - \varepsilon$.

Definition 2.48 (Almost minimizing sequence)**.** Let $\varepsilon_j \searrow 0$. A sequence of surfaces $\{\Sigma^j\}$ in M is ε_j*-almost minimizing* in an open set $U \subset M$ if, for each j, the surface Σ^j is ε_j-almost minimizing in U.

Ideally, we would like our min-max sequence to be *locally minimizing*, as minimal surfaces. However, we cannot ensure this property in general. The best we can expect is to obtain a sequence which is almost minimizing in *small annuli*. We will now define this concept. Given a point $p \in M$ and $0 < r_1 < r_2$, we will denote by

$$A(p, r_1, r_2) := B_{r_2}(p) \setminus \overline{B_{r_1}(p)}$$

the open annulus of inner radius r_1 and outer radius r_2.

Definition 2.49 (Almost minimizing sequence on annuli)**.** Let $\varepsilon_j \searrow 0$. A sequence of surfaces $\{\Sigma^j\}$ is ε_j-almost minimizing in small annuli if for every $p \in M$ there is an $r(p) > 0$ such that for every pair of radii $0 < r_1 < r_2 < r(p)$ the sequence $\{\Sigma^j\}$ is ε_j-almost minimizing in $A(p, r_1, r_2)$ for large j.

We will now prove **Step 3**. Let $\{\Sigma^j_{t_j}\}$ be the min-max sequence which is $\frac{1}{j}$-almost minimizing on annuli and which converges to a stationary varifold Σ^∞. For simplicity, we will denote the min-max sequence by $\{\Sigma^j\} := \{\Sigma^j_{t_j}\}$.

Our goal is to prove that Σ^∞ is a regular minimal surface (possibly with multiplicity). We will show that Σ^∞ satisfies a sufficient condition for being regular, namely the **good replacement condition**; see Theorem 2.53. Let us define first the notion of stable replacement for a varifold.

Definition 2.50 (Stable replacement)**.** Let V be a stationary varifold and $U \subset M$ an open subset. A varifold $\tilde{V}$ is a *stable replacement* for V in U if it verifies that

1. $\tilde{V}$ is stationary,
2. $V|_{M\setminus\bar{U}} = \tilde{V}|_{M\setminus\bar{U}}$,
3. $\|V\| = \|\tilde{V}\|$, and
4. The restriction $\tilde{V}|_U$ is a stable minimal surface Σ with $\bar{\Sigma} \setminus \Sigma \subset \partial U$.

Now we introduce the definition of the good replacement property. Intuitively, a varifold has the good replacement property if it admits sufficiently many (at least three) stable replacements.

Definition 2.51 (Good replacement property)**.** A stationary varifold V has the *good replacement property* in an open subset $U \subset M$ if it satisfies the following conditions.

1. There is a function $r : U \to (0, \infty)$ such that, for every point $x \in U$ and every radii $0 < r_1 < r_2 < r(x)$, the varifold V has a stable replacement V' in the annulus $A(x, r_1, r_2)$.
2. There is a function $r' : U \to (0, \infty)$ such that, for every point $x' \in U$ and every radii $0 < r'_1 < r'_2 < r'(x)$, the varifold V' has a stable replacement V'' in the annulus $A(x', r'_1, r'_2)$.
3. There is a function $r'' : U \to (0, \infty)$ such that, for every point $x'' \in U$ and every radii $0 < r''_1 < r''_2 < r''(x)$, the varifold V'' has a stable replacement V''' in the annulus $A(x'', r''_1, r''_2)$.

We will now prove that stationary limits of min-max sequences which are $\frac{1}{j}$-almost minimizing on annuli have the good replacement property, and so by Theorem 2.53 they are actually minimal surfaces.

Theorem 2.52. *Let $\{\Sigma^j\}$ be a min-max sequence which is $\frac{1}{j}$-almost minimizing on annuli. Suppose that $\{\Sigma^j\}$ converges to a stationary varifold Σ^∞. Then Σ^∞ satisfies the good replacement property.*

Proof. Take a point $p \in \mathrm{supp}(\|\Sigma^\infty\|)$ and consider $A := A(p, r_1, r_2)$ an annulus centered at p in which the sequence $\{\Sigma^j\}$ is $\frac{1}{j}$-almost minimizing. We now fix some j, and consider the following family of isotopies supported in A:

$$\mathsf{Is}_{1/j}(A) = \left\{ \Psi_t \;\middle|\; \mathrm{supp}(\Psi_t) \subset A,\ \|\Psi_t(\Sigma^j)\| \leq \|\Sigma^j\| + \frac{1}{8j} \right\}. \tag{15}$$

Notice that since Σ^j is $\frac{1}{j}$-almost minimizing in A, isotopies $\Psi \in \mathsf{Is}_{1/j}(A)$ cannot decrease the area of Σ^j too much, that is

$$\|\Psi_1(\Sigma^j)\| \geq \|\Sigma^j\| - \frac{1}{j}.$$

Now, for each fixed j, define

$$\kappa_j := \inf\{\|\Psi_1(\Sigma^j)\| \mid \Psi \in \mathsf{Is}_{1/j}(A)\},$$

and consider a sequence of isotopies $\{\Psi^{j,i}\}_i$ belonging to $\mathsf{Is}_{1/j}(A)$ such that the sequence of surfaces $\Sigma^{j,i} := \Psi_1^{j,i}(\Sigma^j)$ satisfies

$$\|\Sigma^{j,i}\| = \|\Psi_1^{j,i}(\Sigma^j)\| \xrightarrow{i\to\infty} \kappa_j.$$

From each surface Σ^j of the sequence, we can produce a limit varifold $\tilde{\Sigma}^j$ as a subsequential limit of the sequence $\Sigma^{j,i}$, that is,

$$\Sigma^{j,i} \xrightharpoonup[i\to\infty]{*} \tilde{\Sigma}^j.$$

This procedure can be viewed as a *constrained minimization problem.* The limit varifolds $\tilde{\Sigma}^j$ satisfy the following properties:

1. As $j \to \infty$, the sequence $\{\tilde{\Sigma}^j\}$ converges in the varifold sense to a stationary integral varifold $\tilde{\Sigma}^\infty$.

2. The new varifolds $\tilde{\Sigma}^j$ satisfy the lower bound $\|\tilde{\Sigma}^j\| \geq \|\Sigma^j\| - \frac{1}{j}$ for every j.

3. They coincide with the original surface Σ^j outside of the annulus A: $\tilde{\Sigma}^j|_{M\setminus A} = \Sigma^j|_{M\setminus A}$.

4. Their restriction $\tilde{\Sigma}^j|_A$ to the annulus A is a stable minimal surface. This is a technical result which can be obtained as a consequence of a result by Meeks-Simon-Yau [MSY82]. More specifically, they proved that when we consider the minimization problem in the set $\mathrm{Is}(A)$ of all isotopies supported in A, then the limit is a stable minimal surface. When we consider $\mathrm{Is}_{1/j}(A)$, the treatment is more involved. We refer the reader to Section 7 of [CDL03] for a complete proof.

We will now show that $\tilde{\Sigma}^\infty$ is a *stable replacement* of the limit Σ^∞ of the original min-max sequence $\{\Sigma^j\}$ in the annulus A, that is, we need to prove that $\tilde{\Sigma}^\infty$ satisfies properties (1)-(4) in Definition 2.50.

Property (2) is immediate, since $\tilde{\Sigma}^j|_{M\setminus A} = \Sigma^j|_{M\setminus A}$. Property (4) is a consequence of the fact that $\tilde{\Sigma}^j$ is a stable minimal surface in A, and so by Corollary 2.12, $\tilde{\Sigma}^\infty|_A$ is also a stable minimal surface. Condition (3) follows from the lower semicontinuity of the mass under varifold convergence applied to the inequalities

$$\|\Sigma^j\| \geq \kappa_j = \|\tilde{\Sigma}^j\| \geq \|\Sigma^j\| - \frac{1}{j}.$$

To prove condition (1), suppose that $\tilde{\Sigma}^\infty$ is not stationary. Then there is a vector field X such that $\delta\tilde{\Sigma}^\infty(X) \neq 0$. By rescaling and manipulating appropriately, one can assume $\delta\tilde{\Sigma}^\infty(X) < -c$, for some $c > 0$. But then there is a subsequence $\{\tilde{\Sigma}^{j'}\}$ of $\{\tilde{\Sigma}^j\}$ such that

$$\delta\tilde{\Sigma}^{j'}(X) < -c/2,$$

for j large enough. It is possible to check that this violates the fact that the surfaces $\{\Sigma^j\}$ are $1/j$-almost-minimizing, leading to a contradiction. Consequently, $\tilde{\Sigma}^\infty$ is a stable replacement for Σ^∞ in A.

So far, we have proved that Σ^∞ admits *one* replacement. However, we can find further replacements by using the following idea: let us define the diagonal sequence $\Gamma^j := \Sigma^{j,i(j)}$, where $i(j)$ is chosen so that $\Gamma^j \rightharpoonup \tilde{\Sigma}^\infty$. One can show that $\{\Gamma^j\}$ itself is a $1/j$-almost minimizing min-max sequence, and so given any small annulus A' we can repeat the arguments used for the original sequence Σ^j. This allows to construct a *further* replacement $\tilde{\tilde{\Sigma}}^\infty$ for $\tilde{\Sigma}^\infty$. We can iterate this argument to obtain a third replacement, showing that the original varifold Σ^∞ has the good replacement property. □

Finally, the fact that the limiting varifold Σ^∞ is a regular minimal surface (possibly disconnected and with multiplicity) is a consequence of the following result.

Theorem 2.53. *Let V be a stationary varifold and $U \subset M$ an open subset. If V has the good replacement property in U, then V is a collection of disjointly embedded regular minimal surfaces possibly with multiplicity.*

We will not prove this fact, and we refer the reader to [CDL03, Proposition 6.3] for a proof. Instead, in order to give an idea of the proof of Theorem 2.53, we will prove the following intermediate regularity result.

Lemma 2.54. *If a varifold V admits replacements in small annuli, then V is integral and every tangent varifold of V is a plane (possibly with multiplicity).*

Proof. We will first prove that V is rectifiable using the rectifiablity criterion (Theorem 2.38). We sketch the idea of the proof:

Let $p \in \operatorname{supp} \|V\|$, which we can assume without loss of generality to be the origin. Let $\lambda_k \searrow 0$ be a sequence, and consider the sequence of dilations $\lambda_k^{-1}V$. Since V is stationary, the sequence $\lambda_k^{-1}V$ converges to a stationary varifold C in $\mathbb{R}^3$ (see Remarks 2.41, 2.42). We notice, by using (12), (13) and dilations, that

$$\Theta(V,p) = \lim_{k\to\infty} \Theta(V,p,\lambda_k) = \lim_{k\to\infty} \frac{\|V\|(B_{\lambda_k}(p))}{\omega_2\lambda_k^2} = \lim_{k\to\infty} \|\lambda_k^{-1}V\|(B_1(p)) = \|C\|(B_1(p)). \quad (16)$$

So, by the rectifiability criterion, we just need to prove that $\|C\|(B_1(p))$ is bounded from below by a positive quantity. The proof is a little bit involved, but the main idea is as follows: given any k, we let $\tilde{V}_k$ be a stable replacement of V in the annulus $A(p, \lambda_k/3, \lambda_k/2)$. We consider now the sequence of varifolds $\lambda_k^{-1}\tilde{V}_k$. Since each term of this sequence is a stationary varifold, it subsequentially converges in the varifold sense to a stationary varifold C'. Moreover, in the annulus $A := A(p, 1/3, 2/3)$, we have that each $\lambda_k^{-1}\tilde{V}_k|_A$ is a stable minimal surface. By Schoen's curvature estimates, the limit $\Gamma := C'|_A$ is also a stable minimal surface (see Corollary 2.12). It is possible to check that the area of Γ must be greater than a certain $c > 0$ independent of p. Hence, by using the definition of replacements,

$$0 < c \leq \|\Gamma\| \leq \|C'\|(A) \leq \|C'\|(B_1(p)) = \|C\|(B_1(p)),$$

so, by (16), V must be rectifiable.

Let us now prove that V is integer rectifiable, that is, integral. First, by Lemma 2.44, C is a cone. On the other hand, the stationary varifold C' satisfies the following properties:

(i) $C' = C$ outside the annulus $A(0, 1/3, 1/2)$.
Indeed, by the properties of the stable replacement we have that, for each k, $V = \tilde{V}_k$ in $\mathbb{R}^3 - A(p, \lambda_k/3, \lambda_k/2)$ for each k, and we conclude the claim by rescaling by λ_k^{-1} and passing to the limit.

(ii) $C'|_A$ is a stable minimal surface, as claimed before.

(iii) $\|C'\|(B(0,r)) = \|C\|(B(0,r))$ for $r \in [0, \frac{1}{3}) \cup (\frac{1}{2}, \infty)$.
For $r \in [0, \frac{1}{3})$, the equality follows directly from property (i). If $r \in (\frac{1}{2}, \infty)$, it is implied again by (i) and the fact that $\|C\|(A) = \|C'\|(A)$

(iv) C' is a stationary rectifiable varifold.

Now, the third property implies that for $r \in (0, \frac{1}{3}) \cup (\frac{1}{2}, \infty)$,

$$\Theta(C', 0, r) = \Theta(C, 0, r) = \Theta(C, 0).$$

But then, by the monotonicity of the density, we conclude that $\Theta(C', 0, r) = \Theta(C, 0)$ for every $r > 0$. Since C' is stationary and rectifiable, it must be a cone, by Lemma 2.43. Therefore, $C' = C$. But since C' is also a smooth minimal surface in $A(0, 1/3, 1/2)$, we conclude that C' is a plane (possibly with integer multiplicity) in $\mathbb{R}^3$, and so is C This implies, in particular, that the density of V at p is an integer, so V is integral. □

Lecture 4

2.6.2 Proof of Step 2

In this lecture we aim to show **Step 2** of the proof of Theorem 1.2.

Theorem 2.55. *There exists a min-max sequence $\{\Sigma^j\} = \{\Sigma^j_{t_j}\}$ that is $\frac{1}{j}$−almost minimizing in annuli.*

It is natural to ask why one should consider the almost-minimizing property on annuli and not in *simpler* open sets like balls. The main reason for this is that the original proof of Theorem 2.55 comes from Almgren-Pitts min-max theory. In that context, they used sweep-outs with point-singularities, which we may avoid by studying the minimization property in annuli.

Another reason why Almgren and Pitts avoided minimization in balls is given by Picture 14. Consider a surface diffeomorphic to $\mathbb{S}^2$ but metrized in such a way that it's a *three-legged starfish.* The picture shows a sweepout by curves with a unique critical slice, which is the figure - eight represented by the green curve. We would expect this geodesic to be minimizing in small balls, and hence almost minimizing. However, this is not the case, at least in Almgren-Pitts formalism: let us consider a ball centered on the red point, where the geodesic self-intersects. According to Almgren-Pitts definition of almost-minimizing, which is a stronger condition than the one we provided here, the geodesic is not almost minimizing in the ball, since one could *transform* it in a pair of curves with less area, (for example, the orange curves on the left). However, this green curve is almost-minimizing in any small annulus: in that case, the curve looks like 4 minimizing segments.

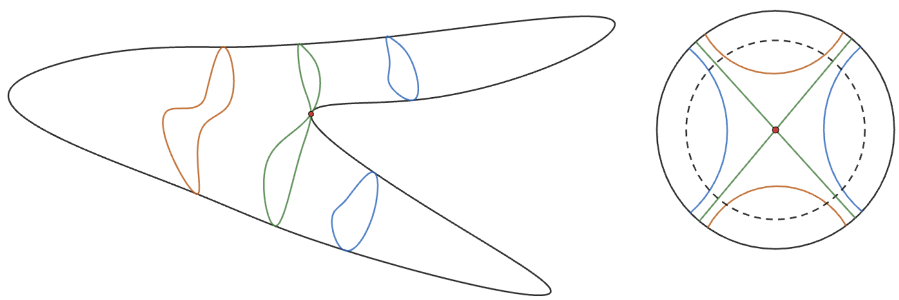

Figure 14: Three legged starfish

The key idea of this result is to define the following concept

Definition 2.56 (Almost-minimizing in pairs)**.** Let (U_1, U_2) be a pair of disjoint open sets in M. We say that a sequence $\{\Sigma^j\}$ is *almost-minimizing* in (U_1, U_2), if Σ^j is $\frac{1}{j}$−almost minimizing in U_1 or U_2.

Now consider the set of pairs

$$\mathcal{A} := \{(U_1, U_2) : d(U_1, U_2) \geq 2 \min(\operatorname{diam} U_1, \operatorname{diam} U_2)\}. \tag{17}$$

One can deduce the following lemma:

Lemma 2.57. *Consider pairs $(U_1, U_2), (V_1, V_2)$ in $\mathcal{A}$. Then, we have $U_i \cap V_j = \varnothing$ for some choice of indices i, j.*

Proposition 2.58. *There exists a min-max sequence Σ^j such that, given any $(U_1, U_2) \in \mathcal{A}$, each Σ^j is $\frac{1}{j}$−almost minimizing in (U_1, U_2).*

This proposition does not prove 2.55 completely, but it follows once we apply 2.58 to the family of pairs $(B_r(p), M \setminus B_{4r}(p)) \in \mathcal{A}$. More precisely, fix $k \in N$ and r such that that $\operatorname{inj}(M) > 4r > 0$, then $(B_r(p), M \setminus B_{4r}(p)) \in \mathcal{A}$. Now by 2.58, Σ^j is $\frac{1}{j}$ almost minimizing either on $B_r(p)$ or $M \setminus B_{4r}(p)$. Thus we have the following 2 cases

1. either Σ^j is almost minimizing on $B_r(x)$ for all x,
2. or there exist x_j and $r_j \to 0$ such that Σ^j is $\frac{1}{j}$ almost minimizing on $M \setminus B_{4r_j}(x_j)$.

If for some $r > 0$, there exists a subsequence of Σ^j satisfying the first case, then we are done. If not, one can find a subsequence of Σ^j which satisfies the second case and such that x_j converges to some $x^* \in M$. Appropriately choosing the $r = r(p)$ function so that we avoid the point x^*, it is possible to conclude Theorem 2.55. We refer the reader to [CDL03] for further details.

Let us now prove Proposition 2.58:

Proof of Proposition 2.58. We work with the minimizing sequence $\{\Sigma_t^j\}$ obtained in Proposition 2.45. After possibly passing to a subsequence, we can assume that

$$\sup_T \left\|\Sigma_t^j\right\| \leq \omega + \frac{1}{10j}.$$

Now, let us fix some J, and take $j \geq J$. We work with large area slices of the form

$$K_j := \left\{t \in [0,1] : |\Sigma_t^j| \geq \omega - \frac{1}{J}\right\}.$$

Proposition 2.58 is a direct consequence of the following:

Claim 2.59. For some $j \geq J$ and $t_0 \in K_j$, the varifold $\Sigma_{t_0}^j$ is $\frac{1}{J}-$almost minimizing in all pairs $(U_1, U_2) \in \mathcal{A}$.

Proof. Suppose not; this would mean that for any $s \in K_j$ and $j \geq J$, there exists a pair $(U_1^s, U_2^s) \in \mathcal{A}$ such that Σ_s^j is not $\frac{1}{J}-$almost minimizing in either of the U_α^s, meaning that there exist isotopies $\{\Psi_{\alpha,t}^s\}$ (for $\alpha = 1, 2$) supported in U_α^s, such that

(i) $\|\Psi_{\alpha,1}^s(\Sigma_s^j)\| \leq \|\Sigma_s^j\| - \frac{1}{J}$.

(ii) $\|\Psi_{\alpha,t}^s(\Sigma_s^j)\| \leq \|\Sigma_s^j\| + \frac{1}{8J}$.

By continuity, we therefore deduce that for each $s \in K_j$, there exists an open set $I_s \subset [0,1]$ about s, such that for all $t_0 \in I_s$, we have a slightly weaker version of the above inequalities:

(i) $\|\Psi_{\alpha,1}^s(\Sigma_{t_0}^j)\| \leq \|\Sigma_{t_0}^j\| - \frac{1}{2J}$.

(ii) $\|\Psi_{\alpha,t}^s(\Sigma_{t_0}^j)\| \leq \|\Sigma_{t_0}^j\| + \frac{1}{4J}$.

For each $s \in K_j$, we consider the corresponding interval I_s. Now, the set $K_j \subset [0,1]$ is **closed**, by continuity, hence compact; this implies that we can choose a finite subcover, and so we can take a finite number of $\{s_k\}$ such that

$$K_j \subset \bigcup_{k=1}^{L} I_{s_k}.$$

After removing or modifying some of the intervals if necessary, one can ensure that each one intersects at most two others, while the covering property is still preserved. In our setting, this means that we can arrange the $\{I_{s_k}\}$ so that each I_{s_k} intersects (at most) $I_{s_{k-1}}$ and $I_{s_{k+1}}$.

By using these intervals and the isotopies associated to each s_k, we will construct now a new sweepout $\{\Gamma_t^j\}$ whose min-max value will be less than the width ω (see Definition 2.20), reaching a contradiction.

Let us consider some interval, for example I_{s_1}, and its associated isotopy $\Psi_{a,t}^{s_1}$. Also, define the *cutoff* function

$$\varphi_{s_1} : I_{s_1} \to [0,1]$$

such that $\varphi_{s_1}(t) \equiv 1$ on some middle sub-interval $\mathcal{I}_{s_1} \subset I_{s_1}$. Using φ_{s_1}, we define a new sweepout by

$$\Gamma_t^j := \Psi^{s_1}_{\alpha,\varphi_{s_1}(t)}(\Sigma_t^j). \tag{18}$$

We will now deal with two cases, depending on whether we can cover K_j with the single interval I_{s_1} or not. First, assume that $K_j \subset I_{s_1}$ and define $\mathcal{I}_{s_1}$ so that $K_j \subset \mathcal{I}_{s_1}$. On this first scenario, $\partial I_{s_1} \cap K_j = \varnothing$. Hence, at this boundary ∂I_{s_1} we have

$$\|\Gamma_t^j\| < \omega - \frac{1}{J}.$$

Using these tools, we use the inequalities (i), (ii) for the isotopies to see that:

$$\begin{aligned}
\text{on } \ \mathcal{I}_{s_1}, \quad \|\Gamma_t^j\| &= \|\Psi^{s_1}_{\alpha,1}(\Sigma_t^j)\| \leq \|\Sigma_t^j\| - \frac{1}{2J} \\
&\leq \omega + \frac{1}{10J} - \frac{1}{2J} \\
&< \omega. \\
\text{on } \ I_{s_1} \setminus \mathcal{I}_{s_1}, \quad \|\tilde{\Sigma}_t^j\| &= \|\Psi^{s_1}_{\alpha,\varphi_{s_1}(t)}(\Sigma_t^j)\| \leq \|\Sigma_t^j\| + \frac{1}{4J} \\
&\leq \omega - \frac{1}{J} + \frac{1}{4J} \\
&< \omega.
\end{aligned}$$

This shows that the sweepout Γ_t^j satisfies $\max \|\Gamma_t^j\| < \omega$, reaching a contradiction.

We will sketch briefly the more general case in which $K_j \subset \bigcup_{k=1}^{L} I_{s_k}$, $L > 1$. The additional difficulty in this case is to study what happens in the overlap between two consecutive intervals $I_{s_k} \cap I_{s_{k+1}}$. For this, we will choose our cutoff functions $\varphi_{s_i} : I_{s_i} \to [0,1]$ (for $i = k, k+1$) such that for all times $\tau \in I_{s_k} \cap I_{s_{k+1}}$, at least one of the $\varphi_{s_i}(\tau)$ is 1 on the overlap and moreover, the sets $U_\alpha^{s_k}, U_\beta^{s_{k+1}}$ in which the isotopies are supported are disjoint. We can always do this by using Lemma 2.57. Under these hypothesis, the surfaces

$$\Gamma_t^j := \Psi^{s_{k+1}}_{\beta,\varphi_{k+1}(t)} \circ \Psi^{s_k}_{\alpha,\varphi_k(t)}(\Sigma_t^j).$$

That is, we apply at the same time both isotopies supported in disjoint open sets. On the overlap of the integrals, we can ensure that, since

$$\|\Sigma_t^j\| \leq \omega + \frac{1}{10J}.$$

and at least one of the isotopies are applied with time 1, we deduce

$$\|\Gamma_t^j\| \leq \|\Sigma_t^j\| - \frac{1}{2J} + \frac{1}{4J} \leq \omega + \frac{1}{10J} - \frac{1}{2J} + \frac{1}{4J} < \omega,$$

as desired. The term $-\frac{1}{2J}$ comes from the fact that at least one of the cuttofs φ_{s_i} is equal to 1. In the worst case, the other cutoff will not be one, and hence we add the term $\frac{1}{4J}$. In this situation, we obtain again a sweepout with $\max \|\Gamma_t^j\| < \omega$, reaching a contradiction and completing the proof of Claim 2.59. □

Remark 2.60. Recall how we defined the saturation, i.e., the homotopy class of sweepouts: we allowed for the possibility of composing with different isotopies at different times; this was precisely in order to allow for the possibility of running the isotopies at different speeds and/or cutting them off.

Remark 2.61. We could also repeat a similar argument to treat the possibility of triple overlaps of intervals, if necessary. For that, we would start in the same way for the first two pair sets U_α^i for $i = 1, 2$; but when considering the third pair of sets U_α^3, we might need to subdivide the intervals further, in case the definition domains of the isotopies U_α^k overlap. From that point on, the same argument as above works.

□

Lecture 5

3 Minimal Heegaard surfaces

It is natural to ask in what other contexts one might apply this min-max construction. We have already showed by min-max arguments that the 3-sphere admits an embedded minimal 2-sphere. What other manifolds can we attain by this procedure?

As we will see, the min-max construction can be used to generalize such a result to compact Riemannian manifolds of positive Ricci curvature. The main difficulty when extending this result to the general case is the definition of the *sweepout* of the manifold. In the case of the 3-sphere, the sweepout was relatively easy to construct by means of vertical 2-spheres. In general, we shall need the notion of *Heegaard splitting* of a 3-manifold, which was introduced by Heegaard in his PhD thesis in 1898.

First, we recall the definition of handlebody.

Definition 3.1 (Handlebodies)**.** A handlebody H is a 3-manifold homeomorphic to the tubular neighbourhood of a one-dimensional graph $\mathcal{G}$ in $\mathbb{R}^3$. The genus of the handlebody H is defined to be $1 - \chi(\mathcal{G})/2$, where $\chi(\mathcal{G})$ denotes the Euler characteristic of $\mathcal{G}$, and it coincides with the genus of the surface ∂H.

Example 3.2. Consider the one-dimensional graph $\mathcal{G}$ in $\mathbb{R}^3$ given by two circles joined by a segment, see Figure 15. The tubular neighbourhood $T_\varepsilon(\mathcal{G})$ of $\mathcal{G}$ is a genus 2 handlebody.

Definition 3.3 (Heegaard surface)**.** A surface $\Sigma^2 \subset M^3$ in a three-manifold is called a *Heegaard surface* if $M^3 \setminus \Sigma^2 = H_1 \sqcup H_2$, where H_1 and H_2 are handlebodies.

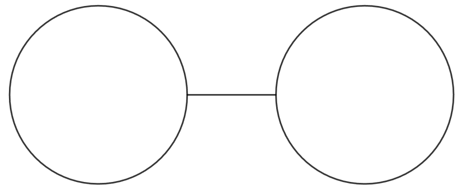

Figure 15: Graph generating the genus 2 handlebody.

From these, the manifold M can be obtained as the gluing $M = H_1 \cup_\varphi H_2$ along their boundaries, where $\varphi : \partial H_1 \to \partial H_2$ is the boundary-identifying homeomorphism.

In the definitions above, we assumed that the surface Σ was orientable. There is also a notion of a non-orientable (*one-sided*) Heegaard splitting that we may need to consider if the ambient manifold is not orientable.

Example 3.4 (Heegaard splitting of the 3-sphere). Consider the manifold $M = \mathbb{S}^3$. Then an embedded 2-sphere Σ in $\mathbb{S}^3$ is a Heegaard surface, since $\mathbb{S}^3 \setminus \Sigma = B_1 \cup B_2$, where B_1 and B_2 are two 3-balls.

Example 3.5 (Heegaard splitting of the real projective space). Consider the real projective space $\mathbb{RP}^3$, which can be defined as follows. Consider the antipodal map $\tau := \tau_{\mathbb{S}^3} : \mathbb{S}^3 \to \mathbb{S}^3$ on the 3-sphere. Then we have an action of the group $\{1, \tau\}$ on $\mathbb{S}^3$, whose quotient is by definition the real projective space,

$$\mathbb{RP}^3 = \mathbb{S}^3/\{1, \tau\}.$$

Since (the closure of) a fundamental domain of the action of the group $\{1, \tau\}$ is a closed hemisphere, hence homeomorphic to the closed 3-ball $\mathbb{B}^3$, an equivalent model for the real projective space is

$$\mathbb{RP}^3 = \mathbb{B}^3/\{1, \tau_{\mathbb{S}^2}\},$$

where $\tau_{\mathbb{S}^2} : \mathbb{S}^2 \to \mathbb{S}^2$ is the antipodal map on $\mathbb{S}^2 = \partial\mathbb{B}^3$.

To obtain a Heegaard splitting of $\mathbb{RP}^3$, consider a cylinder C on $\mathbb{B}^3$ coaxial to the vertical axis; see Figure 16. The antipodal map $\tau_{\mathbb{S}^2}$ on the boundary of $\mathbb{B}^3$ identifies the two circle components of the intersection of the cylinder with the boundary of $\mathbb{B}^3$. Thus, the cylinder C projects to a torus T in $\mathbb{RP}^3$. The projection of $\mathbb{B}^3 \setminus C$ to $\mathbb{RP}^3$ is a disjoint union of two open solid tori. Therefore, T is a Heegaard torus in $\mathbb{RP}^3$.

Proposition 3.6. *Every* $3-$*manifold* M^3 *admits a Heegaard surface.*

Proof. We can consider some triangulation of the manifold M, which always exists; we then let one handlebody to be $H_1 = T_\varepsilon(1 - \mathsf{skeleton})$, that is, a tubular neighborhood of the $1-$skeleton. This construction determines the Heegaard surface Σ uniquely, as $\Sigma = \partial H$; we then obtain $H_2 = M \setminus \overline{H}_1$, which is also a handlebody. $\square$

Proposition 3.6 suggests the following definition.

Definition 3.7 (Heegaard genus). The *Heegaard genus* of a 3-manifold M is defined as the minimal genus over all Heegaard surfaces in M.

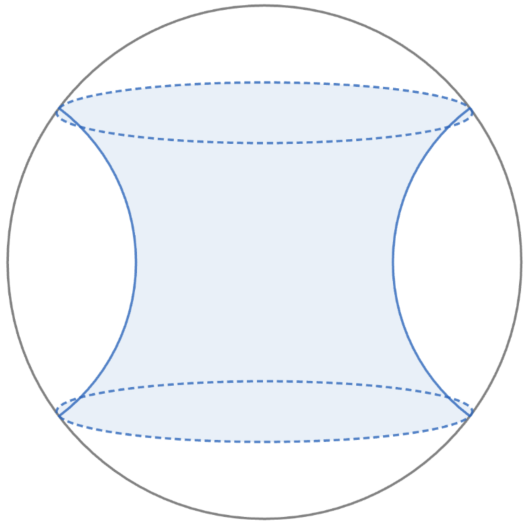

Figure 16: Heegaard torus in $\mathbb{RP}^3$

The min-max technique can be applied in the case of compact manifolds of positive Ricci curvature to prove Theorem 1.3, that we restate here.

Theorem 3.8 (Ketover–Marques–Neves [KMN20])**.** *Let M be a Riemannian 3-manifold of positive Ricci curvature. Then M contains a minimal surface of index 1 realizing the Heegaard genus of M.*

Recall that, by Hamilton's Ricci flow, compact 3-manifolds of positive Ricci curvature are precisely spherical manifolds, that is, finite quotients $\mathbb{S}^3/\Gamma$, where Γ is a finite group acting on $\mathbb{S}^3$ by isometries. In particular, they must be orientable by Synge's theorem. This result raises some interesting questions.

Question 3.9. In the round metric, are the minimal surfaces ensured by Theorem 3.8 unique, up to ambient isometry?

Question 3.10 (Ros). Consider the Poincaré homology sphere S equipped with the round metric. The Poincaré homology sphere has Heegaard genus 2. Hence, by Theorem 3.8, S admits a genus 2 minimal surface Σ. Is Σ isoperimetric?

We will provide a full proof of Theorem 3.8 only in the case $M = \mathbb{RP}^3$. In the rest of this lecture we will present the preliminaries of the proof, which we will state for a general manifold M. In order to apply the min-max procedure to the proof of Theorem 3.8, we need to extend the notion of sweepout to include sweepouts of our manifold by Heegaard surfaces of arbitrary genus.

Definition 3.11. Let (M, g) be any Riemannian manifold, not necessarily of positive Ricci curvature. Suppose M admits a Heegaard splitting of genus g. Then we will consider sweepouts $\{\Sigma_t\}_{t\in[-1,1]}$ such that

1. For each $t \in (-1, 1)$, Σ_t is a Heegaard surface of genus g, and

2. $\Sigma_{\pm 1}$ are homeomorphic to the 1-dimensional spine of a handlebody of genus g.

3. For $t \in (-1, 1)$, $t \mapsto \Sigma_t$ varies *smoothly* in the graphical sense.

4. For $t \in [-1, 1]$, $t \mapsto \Sigma_t$ is continuous in the Hausdorff topology.

As in Section 2.4, we define Π to be the minimal collection of sweepouts containing $\{\Sigma_t\}_t$ which is closed under the production of new sweepouts using isotopies. Recall that if $\{\Sigma_t\}$ is a sweepout of M and Ψ_t is a smooth isotopy on M, then $\{\Psi_t(\Sigma_t)\}_t$ is another sweepout of M. Then, we consider the min-max width of the family Π,

$$\omega := \inf_{\{\Sigma'_t\} \in \Pi} \sup_{t \in [-1,1]} \text{Area}(\Sigma'_t).$$

The same strategy used to prove Theorem 2.21 ensures that $\omega > 0$. Therefore, the application of the min-max argument developed in the previous lectures leads to the following generalization of Theorem 1.2.

Theorem 3.12 (Ketover's genus bound [Ket19])**.** *There exists a min-max sequence $\{\Sigma^j\}$ of surfaces of genus g such that*

$$\Sigma^j \rightharpoonup \sum_{i=1}^{k} m_i \mu^*_{\Gamma_i},$$

where the Γ_i are pairwise disjoint minimal embedded surfaces and $m_i \in \mathbb{N}^*$. The genus of the limiting varifold satisfies

$$\sum_{\Gamma_i \text{ orientable}} m_i \,\text{genus}(\Gamma_i) + \sum_{\Gamma_i \text{ non-orientable}} \frac{m_i}{2} (\text{genus}(\Gamma_i) - 1) \leq g. \tag{19}$$

Marques–Neves derived also a Morse index bound on the limiting minimal surfaces.

Theorem 3.13 (Marques-Neves [MN21])**.** *In this setting, we have*

$$\sum_{i=1}^{k} \text{index}(\Gamma_i) \leq 1.$$

Now we present two important results about minimal surfaces in manifolds of positive Ricci curvature. We shall use these properties in order to rule out certain limiting varifolds that could appear in the min-max procedure.

First, we will show that compact Ricci positive manifolds do not admit orientable stable minimal hypersurfaces.

Proposition 3.14. *Let M be a Riemannian n-manifold of positive Ricci curvature. Then M does not admit an orientable stable minimal hypersurface.*

Proof. Recall that the stability inequality in the manifold M takes on the form

$$-\int_\Sigma \varphi \mathcal{L}_\Sigma \varphi \geq 0$$

for any function $\varphi \in C^\infty_c(\Sigma)$. Here, $\mathcal{L}_\Sigma$ denotes the Jacobi operator

$$\mathcal{L}_\Sigma = \Delta + |A|^2 + \mathsf{Ric}(\nu, \nu)$$

on the ambient manifold M.

Since the manifold M is compact, we can use $\varphi = 1$ as a test function for the stability inequality to obtain

$$\begin{aligned}\int_\Sigma \left(|A|^2 + \mathsf{Ric}(\nu,\nu)\right) &= \int_\Sigma \left(|A|^2 + \mathsf{Ric}(\nu,\nu)\right)\varphi^2 \\ &\le \int_\Sigma |\nabla\varphi|^2 = 0\end{aligned}$$

which gives a contradiction, since $\mathsf{Ric}(\nu,\nu) > 0$. Thus, such a surface Σ cannot exist. □

Remark 3.15. However, it is possible to construct non-orientable stable minimal hypersurfaces in positive Ricci manifolds. For instance, we can consider $\mathbb{RP}^2$, obtained as the image of the equatorial $\mathbb{S}^2 \subset \mathbb{S}^3$ descending onto $\mathbb{RP}^3$ under the identification via the antipodal map. This $\mathbb{RP}^2$ is area-minimizing, hence stable.

If we allow the ambient manifold to be non-orientable, then such stable minimal hypersurfaces may also exist. Notice that there are non-orientable manifolds of dimension $n \ge 4$ which carry metrics of positive Ricci curvature.

Secondly, we will prove that closed minimal hypersurfaces embedded in a compact manifold of positive Ricci curvature satisfy a Frankel property.

Theorem 3.16 (Frankel property [Fra66])**.** *Let M be a complete manifold with positive Ricci curvature $\mathsf{Ric}_M > 0$. Let Σ_1 and Σ_2 be two embedded minimal hypersurfaces in M, and suppose that Σ_1 is compact. Then Σ_1 and Σ_2 must intersect.*

We will give two different proofs of this fact.

Classical proof [Fra66]. Suppose that M admits two embedded minimal hypersurfaces Σ_1 and Σ_2, with Σ_1 compact. Since M is complete and Σ_1 is compact, we can join the two hypersurfaces by a geodesic γ realising the minimal distance between them, see Figure 17.

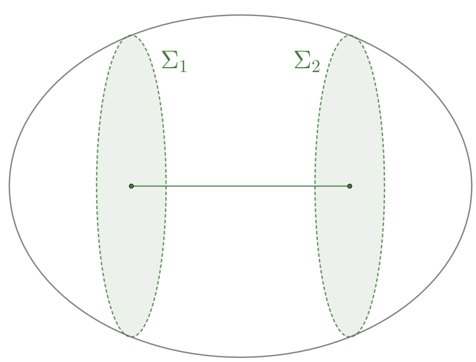

Figure 17: Two embedded minimal hypersurfaces joined by a geodesic realising the distance between them.

The geodesic γ intersects each minimal hypersurface orthogonally. Now we use the formula for the second variation of arc length. The fact that M has positive Ricci curvature implies that there exists a vector field X along γ which is tangent to the minimal hypersurfaces at the endpoints of γ for which the second variation is negative. This contradicts the fact that γ gives the minimal distance bewteen Σ_1 and Σ_2. □

Proof using the mean curvature flow. We give here another proof in the case where M is compact and Σ_i are orientable using the mean curvature flow. We will argue again by contradiction. Suppose that M admits two embedded minimal hypersurfaces Σ_1 and Σ_2 which are oriented. By Proposition 3.14, both Σ_1 and Σ_2 must be unstable.

We know that a mean curvature flow begining at a mean-convex hypersurface Σ either foliates a region bounded by Σ or converges to a stable minimal hypersurface. The hypersurfaces Σ_1 and Σ_2 need not be mean-convex, but we can slightly deform them to a mean-convex hypersurface. More precisely, for each $i = 1, 2$ and a given $\varphi \in \mathcal{C}^\infty(\Sigma_i)$ define

$$\Sigma_{i,t} = \{\exp_p(t\varphi(p)\nu_i(p)) \mid p \in \Sigma_i\},$$

where ν_i is a vector field normal to Σ_i pointing to the region containing the other hypersurface, see Figure 18.

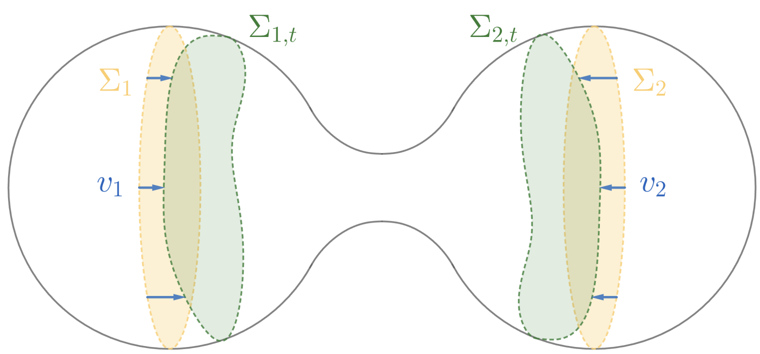

Figure 18: Deformation of the hypersurfaces Σ_1 and Σ_2.

If we take φ to be the eigenfunction of $\mathcal{L}_{\Sigma_i}$ with the lowest eigenvalue $\lambda_i > 0$, mean curvature of $\Sigma_{i,t}$ can be written as

$$H_{\Sigma_{i,t}} = -t\lambda_i\varphi + O(t^2).$$

Since $\varphi > 0$, we have that for a t_i sufficientlly small, the hypersurface Σ_{i,t_i} is mean-convex.

Now we flow Σ_{1,t_1} and Σ_{2,t_2} by the mean curvature flow. Since M does not contain stable minimal hypersurfaces, it is possible to prove that the mean curvature flows starting at Σ_{1,t_1} and Σ_{2,t_2} must intersect. This contradicts the avoidance principle. □

Lecture 6

In this lecture, we aim to give a proof of Theorem 3.8, at least for the particular case $M = \mathbb{RP}^3$. However, we will start by giving a general sketch of the proof for the general case.

Let M be a Riemannian manifold of Heegaard genus g. By Theorem 3.12, there is a min-max sequence $\{\Sigma^j\}$ of surfaces of genus g such that

$$\Sigma^j \rightharpoonup \sum_{i=1}^{k} m_i \mu^*_{\Gamma_i},$$

where Γ_i are disjoint embedded minimal surfaces and $m_i \in \mathbb{N}^*$.

Now, since M has positive Ricci curvature, the Frankel property implies that the min-max sequence $\{\Sigma^j\}$ converges to a unique minimal surface (possibly with multiplicity), that is, $k = 1$. Otherwise, by the Frankel property, two of the limiting minimal surfaces would intersect, which is not possible since the min-max limit has to be embedded.

Suppose first that $\Gamma := \Gamma_1$ is orientable. Since the ambient manifold M is orientable (because it has positive Ricci curvature), then Proposition 3.14 implies that Γ cannot be stable, and by Marques–Neves' index bound (Theorem 3.13), we conclude that Γ has Morse index exactly one. Moreover, Marques–Neves showed that Γ has genus exactly g [MN12, Theorem 3.4].

However, we could also have that the min-max sequence converges to a one-sided multiplicity two minimal Heegaard surface Γ. In order to rule out this case, Ketover–Marques–Neves [KMN20] introduced the catenoid estimate, which we will treat in the following lecture.

For the sake of simplicity, we shall prove the rest of Theorem 3.8 for the case $M = \mathbb{RP}^3$. In this setting, we are actually able to say more:

Theorem 3.17. *The real projective space* $(\mathbb{RP}^3, g)$*, endowed with a metric of positive Ricci curvature, admits an index* 1 *minimal torus* T *satisfying the inequality*

$$|T| < 2 \cdot \inf \left\{ |\Gamma| : \Gamma \text{ embedded } \mathbb{RP}^2 \right\}. \tag{20}$$

Remark 3.18. By the work of Marques–Neves, one may furthermore obtain the **rigidity** of this torus.

Remark 3.19 (Sanity check). We should make sure that the result we are trying to prove is actually true! For this, we can perform a sanity check in the case of "round" $\mathbb{RP}^3$, i.e., descending from the sphere $\mathbb{S}^3$. Let us consider the Clifford torus $C \subset \mathbb{S}^3$; this descends into a torus in $\mathbb{RP}^3$ that realizes the Heegaard genus, hence should satisfy the result. Indeed, we can check that the Clifford torus has area $|C| = 2\pi^2$, hence its projected image has $|\pi(C)| = \pi^2$ in $\mathbb{RP}^3$.

Likewise, let us consider the "equatorial" $\mathbb{RP}^2$ embedded in $\mathbb{RP}^3$, which is the equatorial $\mathbb{S}^2 \subset \mathbb{S}^3$ descending under the projection map π. Since the $\mathbb{S}^2$ has area 4π, it follows that the embedded $\mathbb{RP}^3$ as area 2π.

Comparing the two, we deduce that

$$\pi^2 = |\pi(C)| < 4\pi = 2 \cdot 2\pi = 2|\pi(\mathbb{S}^2)|$$

and so the desired inequality (20) holds.

Proof of Theorem 3.17. Let us perform a min-max procedure involving a Heegaard sweepout of $(\mathbb{RP}^3, g)$. By the arguments above, the min-max procedure converges to a single embedded minimal surface Σ of multiplicity m. By the classification of surfaces, Ketover's genus bounds in Theorem 3.12, and since non-orientable limits must appear with even multiplicity, the only possibilities are:

(i) $\Sigma \simeq \mathbb{S}^2$, for any multiplicity $m \geq 1$;

(ii) $\Sigma \simeq \mathbb{T}^2$ and $m = 1$;

(iii) $\Sigma \simeq \mathbb{RP}^2$, for any even multiplicity $m \geq 2$;

(iv) $\Sigma \simeq \mathbb{RP}^2 \# \mathbb{RP}^2$ with even multiplicity.

The first and last cases can immediately be ruled out.

- Suppose Σ is homeomorphic to a 2-sphere. By the Frankel property, the lift $p^{-1}(\Sigma)$ of the surface Σ to the universal cover $\mathbb{S}^3$ must be connected. Thus $p^{-1}(\Sigma)$ is homeomorphic to a 2-sphere. The Riemann-Hurwitz formula for the covering $p^{-1}(\Sigma) \to \Sigma$ can be written as follows:
$$\chi(p^{-1}(\Sigma)) = 2\chi(\Sigma),$$
which gives a contradiction, since $\chi(p^{-1}(\Sigma)) = \chi(\Sigma) = 2$.
- For topological reasons, the Klein bottle $\mathbb{RP}^2 \# \mathbb{RP}^2$ cannot be embedded in the real projective space $\mathbb{RP}^3$. More generally, any non-orientable surface embedded in $\mathbb{RP}^3$ must have odd Euler characteristic [BW69].

Our goal is to show that Σ is precisely the torus occurring with multiplicity 1. Thus, it suffices to rule out the possibility of having an embedded $\mathbb{RP}^2$ of even multiplicity $m \geq 2$. In order to do so, we will show the following:

Lemma 3.20. *There is a sweepout by Heegaard tori whose associated* ***min-max width*** ω *satisfies*
$$\omega < 2 \inf \left\{ |\Gamma| : \Gamma \text{ embedded } \mathbb{RP}^2 \right\}. \tag{21}$$

Proof of Lemma 3.20. First, we show that the infimum of (21) is actually attained by a minimal real projective plane Γ. This surface can be obtained as a consequence of the result by Meeks-Simon-Yau [MSY82, Theorem 1]: indeed, let us consider the area minimization problem on the set of 2-spheres in $\mathbb{RP}^3$. We consider a minimizing sequence of spheres, which by Meeks-Simon-Yau must converge to a surface Γ which is either a minimal, stable sphere or a minimal real projective plane Γ of least area. The limit cannot be stable since $\mathrm{Ric} > 0$ (see Proposition 3.14), so Γ must be a real projective plane. Thus, to prove inequality (21) it suffices to check that there exists a sweepout by tori in $\mathbb{RP}^3$ whose maximum area is less than $2|\Gamma|$. We will devote the rest of this Lecture to construct such a sweepout, proving Lemma 3.20. □

Once the proof of Lemma 3.20 is completed, we deduce from Equation (21) that the min-max limit Σ cannot be a real projective plane with multiplicity 2, so Theorem 3.17 holds. □

Lemma 3.21. *Let Γ be a projective plane in $(\mathbb{RP}^3, g)$ with least area. Then, the complement $\mathbb{RP}^3 \setminus \Gamma$ is a 3−ball.*

Sketch of the proof. Let us lift the minimizing projective plane Γ onto the double-cover sphere $\mathbb{S}^3$; we denote the corresponding double-cover by $\tilde{\Gamma} := \pi^{-1}(\Gamma)$. The Euler characteristic of $\tilde{\Gamma}$ has to be 2, so $\tilde{\Gamma}$ is a sphere. Moreover, $\tilde{\Gamma}$ has to be minimal and embedded. One can show as in [MSY82, Corollary to Theorem 5] that $\tilde{\Gamma}$ is a Heegaard surface of genus zero, and so $\mathbb{S}^3 \setminus \tilde{\Gamma}$ consists of two balls. Consequently, the projection of these two balls in $\mathbb{RP}^3$, which coincides with $\mathbb{RP}^3 \setminus \Gamma$, has to be a single ball. $\square$

We will now construct a sweepout by tori in $\mathbb{RP}^3$ whose maximum area will be less than $2P$. To do so, we will first construct a certain sweepout in $\mathbb{S}^3$ which we will later project in $\mathbb{RP}^3$.

As before, let Γ be a real projective plane of least area, and set $P := |\Gamma|$. Denote by $\tilde{\Gamma} \subset \mathbb{S}^3$ the corresponding lift sphere in $\mathbb{S}^3$, which satisfies $2P = |\tilde{\Gamma}|$. Notice that $\tilde{\Gamma}$ is minimal, and so it has to be **unstable**: indeed, the condition of positive Ricci curvature in $\mathbb{RP}^3$ is preserved upon passing to its cover. Now, by Lemma 3.21, $\tilde{\Gamma}$ divides $\mathbb{S}^3$ into two balls B_1, B_2, which are related by the involution map in $\mathbb{S}^3$. We will now define a *sweepout* by spheres in B_1. Let us detail this construction: since $\tilde{\Gamma}$ is unstable, there exists an eigenfunction $\phi > 0$ associated to the lowest eigenvalue of the Jacobi operator of $\tilde{\Gamma}$. Given some small $t_0 > 0$, we can define the slices

$$\tilde{\Gamma}_t = \left\{ \exp_p(t\phi(p)N(p)), p \in \tilde{\Gamma} \right\}, \tag{22}$$

for $t \in [0, t_0]$, where N denotes the unit normal to $\tilde{\Gamma} = \tilde{\Gamma}_0$ pointing to the interior of B_1. Denoting by $\lambda < 0$ the eigenvalue associated to Γ, it follows

$$\begin{aligned} |\tilde{\Gamma_t}| &= |\tilde{\Gamma}| - t\int H\phi + t^2 \int \phi L\phi + O(t^3) \\ &= |\tilde{\Gamma}| + \lambda t^2 \int \phi^2 + o(t^3) \end{aligned}$$

As $\lambda < 0$, we deduce that $|\tilde{\Gamma_t}| = 2P - ct^2$, $0 \leq t \leq t_0$ for some positive constant c. Also one can see that the mean curvature of the slices $H_{\tilde{\Gamma}_t}$ has to be strictly positive, for $0 < t \leq t_0$.

Let $R_{t_0} := B_1 \setminus \bigcup_{t=0}^{t=t_0} \tilde{\Gamma}_t$, and consider the family of sweepouts given by

$$\Pi = \left\{ \text{sweepouts } \{\Sigma_t\}_{t=t_0}^1 \text{ in } \ R_{t_0} \ \text{ s.t. } \ \Sigma_{t_0} = \tilde{\Gamma}_{t_0} \text{ and } \ \Sigma_1 = \{\text{pt}\} \right\}.$$

Let w denote the *min-max width* of Π. Notice that $w \leq |\tilde{\Gamma_{t_0}}|$: otherwise, one could consider a min-max procedure with boundary as in Theorem 2.1 of [MN12], producing a minimal 2-sphere in the interior of R_{t_0}. This is not possible, as it would contradict the Frankel property. In fact, it is possible to find an extension of the slices $\tilde{\Gamma}_t$ for $t > t_0$ by a sweepout in Π satisfying

$$|\tilde{\Gamma}_t| \leq 2P - ct^2 \tag{23}$$

for all $0 \leq t \leq 1$ and some $c > 0$. Now, let $\Sigma_t := \pi\left(\tilde{\Gamma}_t\right)$ denote the projection of the spheres $\tilde{\Gamma}_t$ in $\mathbb{RP}^3$; see Figure 19. It is possible to check that $|\Sigma_t| = |\tilde{\Gamma}_t|$. From Σ_t we will now construct a new sweepout by tori $\tilde{\Sigma}_t$. The idea is as follows: let $\varepsilon > 0$ be a small constant.

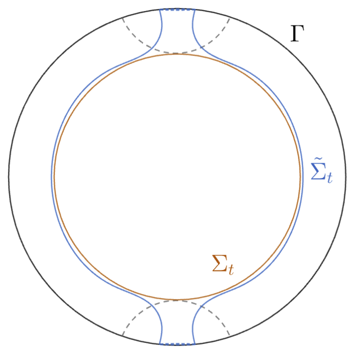

Figure 19: Representation of $\mathbb{RP}^3$ as a ball. The minimal projective plane Γ lies in *the boundary* of the ball, and the spheres Σ_t lie in the interior of the ball. We also represent the tori $\tilde{\Sigma}_t$.

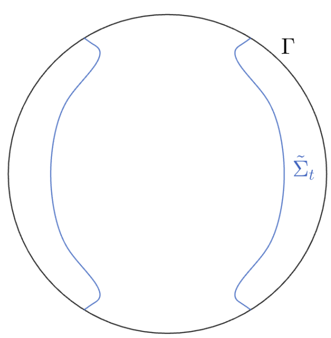

Figure 20: Opening the neck for t near zero on $\tilde{\Sigma}_t$.

For $t > \varepsilon$, $\tilde{\Sigma}_t$ will be the result of attaching a *thin neck* that connects Σ_t to itself as in Figure 19. Thus, $\tilde{\Sigma}_t$ are **tori**. If the necks are thin enough, we can ensure that $\tilde{\Sigma}_t$ also satisfy (23).

We now want to extend the tori $\tilde{\Sigma}_t$ for $t < \varepsilon$. Moreover, we aim to control the area of these tori so that $|\tilde{\Sigma}_t|$ is strictly less than $2P$ for all t. The idea is as follows: as t approaches zero, the spheres Σ_t approach Γ with multiplicity 2; see Figure 19. To avoid that increase in area, we *open* the neck as in Figure 20 as t approaches zero. Intuitively, if the surfaces Σ_t are close enough to Γ and we open the necks appropriately, then the area of $\tilde{\Sigma}_t$ will be *much less* than that of Σ_t, proving the bound

$$|\tilde{\Sigma}_t| < 2P - ct^2 \leq 2P$$

for all t, from which we conclude Lemma 3.20.

This *neck opening* process will be achieved by a **catenoid estimate** method. We will devote the next Lecture to present this technique.

3.1 The catenoid estimate

We recall that the catenoid is a rotational minimal surface in $\mathbb{R}^3$ first described by Euler (1744). Up to rescaling, it is described by the parameter a, which controls its inner neck. Its equation is then given by

$$r = a\cosh\left(\frac{z}{a}\right), \qquad \text{for} \;\; a \in \mathbb{R}^+, \tag{24}$$

where $r = \sqrt{x^2+y^2}$. The catenoid appears as the solution to the following Plateau's problem in $\mathbb{R}^3$: consider a pair of parallel circumferences of radius r and separated by height h; assume that $h \ll r$ and denote this Plateau data by $\Gamma(h, r)$.

Question 3.22. What are the minimal surfaces bounded by $\Gamma(r, h)$?

A trivial solution to this question is the configuration given by the two disks bounded by these two circumferences. This is a stable (disconnected) minimal surface, which we may denote by $D(r, h)$; its area is then $|D(r, h)| = 2\pi r^2$. However, we also may *glue* a rotational neck connecting these disks, obtaining a catenoid.

Lecture 7

We will now present the main ideas behind the catenoid estimate procedure. As an application, we will study the construction of the sweepout by tori $\tilde{\Sigma}_t \subset \mathbb{RP}^3$ defined in the previous section. For a more detailed exposition, we refer the reader to [KMN20]. We state the next general result:

Theorem 3.23. *Let (M, g) be a Riemannian 3-manifold and let Σ be a closed orientable unstable embedded minimal surface of genus g in M. Denote by ϕ the lowest eigenfunction of the Jacobi operator on Σ, and let ν be a choice of unit normal on Σ. Fix $p_1, \ldots, p_k \in \Sigma$ and a graph $\mathcal{G}$ in Σ so that $\Sigma \setminus \{p_1, \cdots, p_k\}$ retracts to $\mathcal{G}$. Then there exist $\varepsilon_0 > 0$ and $\tau > 0$ so that whenever $\varepsilon \leq \varepsilon_0$, there exist a sweepout $\{\Lambda_s\}_{s=0}^{\varepsilon}$ of a tubular neighborhood $T_{\varepsilon\phi}(\Sigma)$ so that:*

- *Λ_s is a smooth surface of genus $2g+k-1$ (i.e. two copies of Σ joined by k necks) for $0 < s < 1$,*

- *$\Lambda_0 = \mathcal{G}$,*

- *$\Lambda_\varepsilon = \{\exp_p(\pm\varepsilon\phi(p)\nu(p)), p \in \Sigma\} \cup \left(\bigcup_{i=1}^k \{\exp_{p_i}(\lambda\phi(p_i)\nu(p_i)), \lambda \in [-\varepsilon, \varepsilon]\}\right)$ (i.e, two surfaces which are deformations of Σ joined by k segments or 'thin necks'),*

- *For all $0 \le s \le 1$, $|\Lambda_s| \le 2|\Sigma| - \tau\epsilon^2 < 2|\Sigma|$.*

Before going into the details of the proof of Theorem 3.23, let us apply this result to our situation. We wanted to complete the sweepout by tori $\tilde{\Sigma}_t \subset \mathbb{RP}^3$ for $t < \varepsilon$. So, first of all, we lift the tori $\tilde{\Sigma}_t$ to $\mathbb{S}^3$, and denote them by $\tilde{\Lambda}_t$. For $t = \varepsilon$, the situation looks like Figure 21: the tori $\tilde{\Lambda}_t$ are cut in half by the minimal 2-sphere $\tilde{\Gamma}$, and they seem like two balls joined by two thin necks, which we would like to open for $t < \varepsilon$. Let us apply Theorem 3.23 in our situation, where we consider:

- The ambient space M will be the 3-sphere $\mathbb{S}^3$.

- The closed orientable surface $\Sigma \subset \mathbb{S}^3$ will be the minimal 2-sphere $\tilde{\Gamma}$. Hence, $g = 0$.

- $k = 2$, as we have two necks. By the symmetry of the construction, the points p_1, p_2 (and their respective necks) have to be related by the involution of $\mathbb{S}^3$ which defines the quotient $\mathbb{RP}^3$.

- $\mathcal{G}$ will be a circumference in the 2-sphere $\tilde{\Gamma}$.

- The genus $2 \cdot 0 + 2 - 1 = 1$ surface Λ_ε which appears in Theorem 3.23 corresponds to the torus $\tilde{\Lambda}_\varepsilon$ with *thin necks* that we have defined in $\mathbb{S}^3$.

Consequently, Theorem 3.23 helps us extend the sweepout $\tilde{\Lambda}_t$ for $t < \varepsilon$ in a way such that $|\tilde{\Lambda}_t| < 2|\tilde{\Gamma}| = 4P$. Since the tori $\tilde{\Lambda}_t$ have twice the area of $\tilde{\Sigma}_t \subset \mathbb{RP}^3$, we deduce that the bound

$$|\tilde{\Sigma}_t| < 2P$$

holds for all $t \in [0, 1]$, proving Theorem 3.17.

Sketch of the proof of Theorem 3.23. We sketch the case $k = 1$, the general case follows with some direct modifications. Let $p \in \Sigma$. One of the key ingredients to prove the catenoid estimate is the "log cutoff trick". This construction was used by Choi and Schoen, together with the Choi–Wang first eigenvalue estimate.

For this end, we consider small geodesic balls B_R centered at p and define the function

$$\eta_R(q) := \begin{cases} 1, & \text{on } \Sigma \setminus B_R, \\ 0, & \text{on } B_{R^2}, \\ \frac{\log(r/R^2)}{\log(1/R)}, \text{on } R^2 \le r \le R & \end{cases}$$

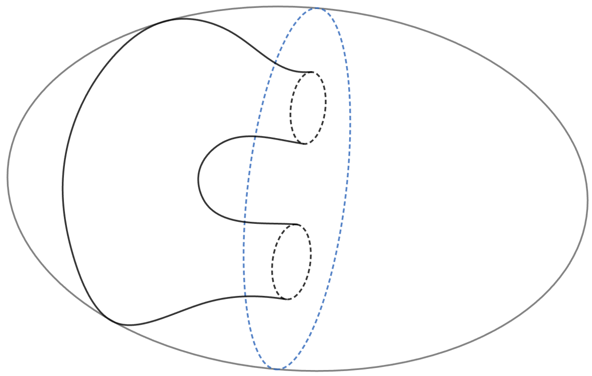

Figure 21: Half torus in $\mathbb{S}^3$. The torus is split by the minimal 2-sphere $\tilde{\Gamma} \subset$ which divides $\mathbb{S}^3$ into two equivariant spheres.

where $r = d(p, q)$ denotes the geodesic distance from p to q. Now, let $\phi_R(q) = \phi(q)\eta_R(q)$. If we assume that R and q are small enough, so that the neighbourhood of p is *almost Euclidean*, the gradient is approximately given by

$$|\nabla \eta_R| = \left| \frac{1}{-\log R} \nabla \left(\log r - \log R^2 \right) \right| = \left| \frac{1}{r} \frac{1}{-\log R} \chi_{B_R \setminus B_{R^2}} \right|$$

Hence, we obtain

$$|\nabla \eta_R|^2 = \frac{1}{r^2} \frac{1}{(\log R)^2}.$$

It follows that

$$\int |\nabla \eta_R|^2 = \int_{R^2}^{R} \frac{1}{r^2} \frac{1}{(\log R)^2} r \, dr \, 2\pi$$

which is **finite** and goes to 0 as $R \to \infty$.

Using these log-cutoff functions, we consider the parallel surface $\hat{\Sigma}_{h,R}$ of the form

$$\hat{\Sigma}_{\pm h,R} = \left\{ \exp_q(\pm h\nu(q)\phi_R(q)) \;:\; q \in \Sigma \right\}$$

by analogy to the construction (22). We also define

$$\Lambda_{h,R} = (\hat{\Sigma}_{+h,R} \setminus B_{R^2}) \cup (\hat{\Sigma}_{-h,R} \setminus B_{R^2})$$

On a neighbourhood of p, $\Lambda_{h,R}$ looks like a *catenoid* with a *neck* of radius R^2 centered around p; see Figure 22. The *height* of the *catenoid* is controlled by h. When $R \mapsto 0$, this surface converges in the varifold sense to

$$\left\{ \exp_q(+h\nu(q)\phi(q)) \;:\; q \in \Sigma \right\} \cup \left\{ \exp_q(-h\nu(q)\phi(q)) \;:\; q \in \Sigma \right\}$$

Intuitively, in this limit we reduce the *radius of the neck* until it dissapears, so we end up with two disks on a neighbourhood of p.

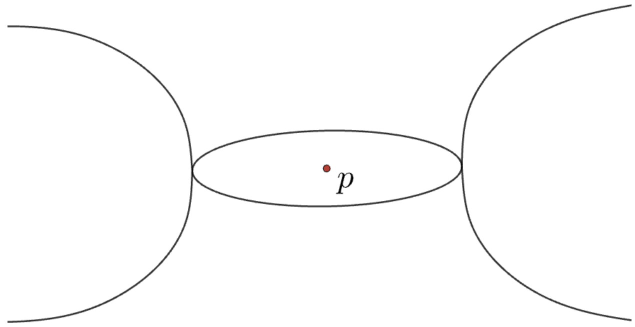

Figure 22: Surface $\Lambda_{h,R}$ on a neighbourhood of p.

Using the Taylor expansion of the area functional, we compute that the area of this surface is approximately

$$\begin{aligned}
|\hat{\Sigma}_{h,R}| &= |\Sigma| - h^2 \int_\Sigma \phi_R \mathcal{L}_\Sigma \phi_R + O(h^3), \qquad \text{since } |\Sigma| \text{ is minimal,} \\
&= |\Sigma| - h^2 \int_\Sigma \phi_R \left(\Delta_\Sigma \phi_R + |A|^2 \phi_R + \mathsf{Ric}(\nu,\nu)\phi_R \right) + O(h^3) \\
&= |\Sigma| + h^2 \int_\Sigma |\nabla \phi_R|^2 - |A|^2 \phi_R^2 - \mathsf{Ric}(\nu,\nu)\phi_R^2 + O(h^3)
\end{aligned}$$

using integration by parts. We can express this alternatively as

$$|\hat{\Sigma}_{h,R}| = |\Sigma| + h^2 \int_{B_R \setminus B_{R^2}} |\nabla \phi_R|^2 + h^2 \int_{(\Sigma \setminus B_R) \cup B_{R^2}} |\nabla \phi_R|^2 - h^2 \int_\Sigma |A|^2 \phi_R^2 + \mathsf{Ric}(\nu,\nu)\phi_R^2 + O(h^3).$$

First, we notice that $|\nabla \phi_R|^2$ is exactly zero on B_{R^2} and $\Sigma \setminus B_R$ Moreover, in $B_R \setminus B_{R^2}$,

$$\int_{B_R \setminus B_{R^2}} |\nabla \eta_R|^2 = \frac{1}{(\log R)^2} \log r \Big|_{R^2}^{R} \leq \frac{C}{-\log R}.$$

In particular, this implies that $\int_{B_1} |\nabla \phi_R|^2 \to 0$ as $R \to 0^+$, so the integral on $|\nabla \phi_R|$ can be bounded by $\frac{C}{-\log R}$ for small R and some constant C. On the other hand, we can bound the rest of the terms in the integral by $-\frac{|\lambda|}{2}h^2$, where λ denotes the eigenvalue associated to ϕ. It follows that

$$|\hat{\Sigma}_{h,R}| \leq |\Sigma| + \frac{Ch^2}{-\log R} - \frac{|\lambda|}{2}h^2.$$

Thus adding the contribution from the two components of $\Lambda_{h,R}$ we get

$$|\Lambda_{h,R}| \leq 2|\Sigma| + \frac{2Ch^2}{-\log R} - |\lambda| h^2 - 2|B_{R^2}|$$

Finally, we can choose R_0 small, so that $\frac{2C}{-\log R} < \frac{|\lambda|}{2}$ for all $R \leq R_0$, to ensure that

$$|\Lambda_{h,R}| \leq 2|\Sigma| - \frac{|\lambda|}{2}h^2 - 2|B_{R^2}| < 2|\Sigma| - \frac{|\lambda|}{2}h^2 \tag{25}$$

Now we fix the parameter $R = R_0$ and we analyze what happens in the limit $h \to 0$. In this situation, Λ_{h,R_0} converges to $\Sigma \setminus B_{R_0}$ with multiplicity two. By assumption $\Sigma \setminus B_{R_0}$ retracts to a graph $\mathcal{G}$. We will now see how to deform Λ_{h,R_0} to $\Sigma \setminus B_{R_0}$ (and to $\mathcal{G}$).

For $s \in [0,1]$, take a real non-negative smooth function $h : [0,1] \to [0,h]$, satisfying $h(0) = 0$ and $h(1) = h_0$. Define

$$\Lambda_{h(s),R_0} = \{\exp_q (\pm h(s)\phi_{R_0}(q)\nu(q)), q \in \Sigma\}$$

We can show that the areas are controlled by $|\Lambda_{h(s),R_0}| \leq 2|\Sigma| - \frac{|\lambda|}{2}h(s)^2 - 2|B_{{R_0}^2}|$. With this in mind, consider the sweepout given by

$$\Lambda_s = \begin{cases} \Lambda_{h(2s),R_0}, & 0 \leq s \leq \frac{1}{2} \\ \Lambda_{h_0,2(1-s)R_0}, & \frac{1}{2} \leq s \leq 1. \end{cases}$$

For $s = 0$, Λ_0 is two copies of $\Sigma \setminus B_{R_0}$. For $0 < s < 1/2$, we *lift* the height of the surface, creating a catenoid with *neck* R_0. For $1/2 < s < 1$, we make the neck smaller, so that it disappears for $s = 1$. One can check that the area of Λ_s always strictly less than $2|\Sigma|$. It is possible to extend this family of surfaces for $s \in [-1,0]$ so that we deform Λ_0 into $\Lambda_{-1} := \mathcal{G}$. This extension also respects the desired area bound. Hence, up to an appropriate rescaling in the variable s, we deduce Theorem 3.23. □

4 Multiple minimal 2-spheres

Our motivation to introduce the min-max procedure was to find a minimal 2-sphere in any Riemannian 3-sphere $(\mathbb{S}^3, g)$. Such a procedure can be unterstood as carrying out a Morse theoretical study of the space of closed embedded 2-spheres for the area functional, for which minimal surfaces are critical points. Now we will see how to extend these arguments in order to detect **additional** critical points of the area functional, that is, different embedded minimal 2-spheres in $(\mathbb{S}^3, g)$.

In his problem section, Yau conjectured that any Riemannian manifold diffeomorphic to the 3-sphere contains four embedded minimal 2-spheres.

Conjecture 4.1 (Yau's 89th problem [Yau82])**.** *Any Riemannian 3-sphere contains (at least) four distinct minimal spheres.*

The first motivation for this conjecture comes from the example of the 3-dimensional ellipsoid.

Example 4.2. Consider the ellipsoid $E(a,b,c,d)$ of semiaxis $a > b > c > d > 0$ in $\mathbb{R}^4$,

$$E(a,b,c,d) := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} + \frac{x_4^2}{d^2} = 1\right\}.$$

The ellipsoid $E(a,b,c,d)$ contains four minimal 2-spheres, namely the four planar 2-spheres, that is, the 2-spheres given by the intersection of $E(a,b,c,d)$ with the coordinate hyperplanes,

$$S_i := E(a,b,c,d) \cap \{x_i = 0\}$$

Actually, Yau also conjectured that the only minimal 2-spheres contained in the ellipsoid $E(a,b,c,d)$ are the planar spheres.

Conjecture 4.3 (Yau [Yau19])**.** *The only minimal 2-spheres contained in the ellipsoid $E(a,b,c,d)$ are planar.*

Almgren proved that the only immersed minimal 2-spheres in the round 3-sphere are the planar ones. Later, White proved that if the values of the semiaxis a,b,c,d are close enough, then Yau's conjecture is true.

The second fact that motivates Conjecture 4.1 comes from the topology of the space of embedded 2-spheres in $\mathbb{S}^3$.

Theorem 4.4 (Smale's conjecture, proven by Hatcher [Hat83])**.** *The space of embbeded 2-spheres in $\mathbb{S}^3$ deformation retracts to the space of equators.*

Notice that the space of equators of $\mathbb{S}^3$ is homeomorphic to $\mathbb{RP}^3$. Hence, since a Morse function on $\mathbb{RP}^3$ has four nontrivial critical points, a Riemannian 3-sphere should contain four distinct minimal 2-spheres. We will deal with a generalized version of Smale's conjecture in Section 5.

Haslhofer–Ketover gave a partial answer to Yau's conjecture. They proved that for a generic class of Riemannian metrics on $\mathbb{S}^3$, namely bumpy metrics on $\mathbb{S}^3$, one can always find two distinct minimal 2-spheres. A metric is bumpy if every closed embedded minimal surface is non-degenerate.

Definition 4.5 (Bumpy metric)**.** A Riemannian metric g on $\mathbb{S}^3$ is *bumpy* if given any embedded minimal surface Σ, the only function satisfying $\mathcal{L}_\Sigma \varphi = 0$ is $\varphi \equiv 0$.

Theorem 4.6 (Haslhofer–Ketover [HK19])**.** *Let g be a bumpy Riemannian metric on $\mathbb{S}^3$. Then $(\mathbb{S}^3 g)$ contains at least two distinct minimal 2-spheres.*

It is worth mentioning that Haslhofer–Ketover's result was recently improved.

Theorem 4.7 (Wang–Zhou[WZ24])**.** *Let g be a bumpy Riemannian metric on $\mathbb{S}^3$. Then $(\mathbb{S}^3 g)$ contains at least four distinct minimal 2-spheres.*

Sketch of the proof of Theorem 4.6. Let $(\mathbb{S}^3, g)$ be a bumpy Riemannian 3-sphere. By Theorem 1.2, there exists an embedded minimal 2-sphere Γ_1 in $(\mathbb{S}^3, g)$.

Suppose first that Γ_1 is stable. Then the shape of the manifold M is similar to a dumbell. If we split $(\mathbb{S}^3, g)$ along Γ_1 into two 3-balls, then we can sweepout each half by a 1-parameter family of embedded 2-spheres. By [KLS19], each 3-ball contains an unstable minimal 2-sphere of Morse index 1. In total, $(\mathbb{S}^3, g)$ contains three minimal 2-spheres.

Now, suppose that the 2-sphere Γ_1 is unstable. By the index bound from Theorem 3.13, Γ_1 has Morse index 1. Actually, one can see that Γ_1 is the embedded minimal 2-sphere of least area. In this case, the idea consists in constructing another sweepout of $(\mathbb{S}^3, g)$ by 2-spheres which converges to a minimal 2-sphere Γ_2 distinct from Γ_1.

In order to define such a sweepout, start by pushing Γ_1 away a bit to both sides using the lowest eigenfunction of the stability operator, in order to produce two mean-convex 2-spheres Γ_1' and Γ_1''. Now, we can flow each of this surfaces by the mean curvature flow. For the sake

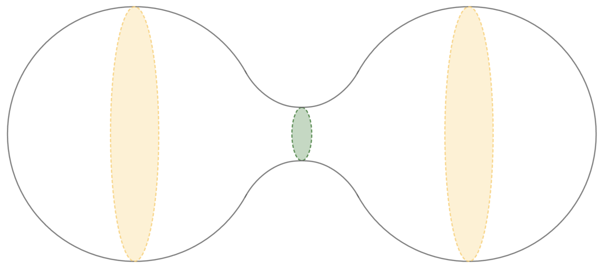

Figure 23: The Riemannian 3-sphere contains one stable minimal 2-sphere and two unstable minimal 2-spheres of index 1.

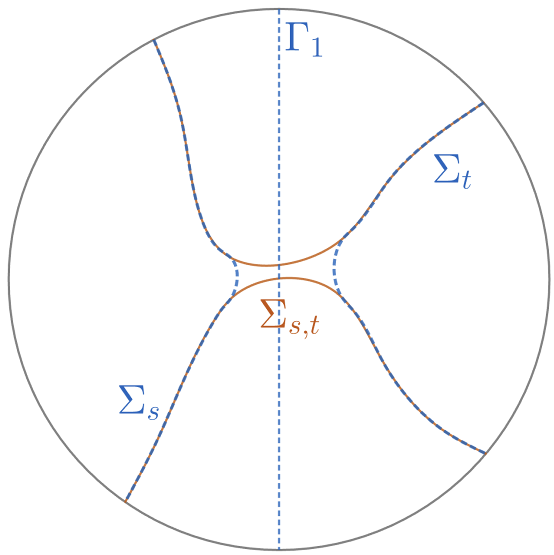

Figure 24: Longitudinal section of a sweepout of the 3-sphere by the 2-parametric family of spheres $\{\Sigma_{s,t}\}$.

of simplicity, suppose that the mean curvature flow remains smooth. Then, the evolution of the mean curvature flow can lead to two different situations. If the mean curvature flow is defined for all times, then the evolving surface converges to a minimal embedded 2-sphere, which would be different from the original Γ_1 since it is contained in a region which is disjoint from it.

Hence, suppose that the mean curvature flow for both surfaces extincts in finite time, this is, the surfaces converge to a point by the mean curvature flow. Then we obtain a foliation of each 3-ball bounded by Γ_1' and Γ_1'' by mean-convex embedded 2-spheres. From these foliations we can construct an *optimal foliation*, that is, a foliation $\{\Sigma_t\}_{t\in[-1,1]}$ such that $\Sigma_0 = \Gamma_1$ and satisfying

$$\|\Sigma_t\| \leq \|\Sigma_0\| - ct^2 = \|\Gamma_1\| - ct^2, \tag{26}$$

for all $t \in [-1,1]$.

From the optimal foliation $\{\Sigma_t\}_{t\in[-1,1]}$, we construct a 2-parameter sweepout $\{\Sigma_{s,t}\}_{s,t\in[-1,1]}$ as follows. For each $s,t \in [-1,1]$ with $s < t$, define $\Sigma_{s,t}$ to be the surface obtained from Σ_s and Σ_t by removing a 2-disc from each, and then gluing a very thin tube T, see Figure 24.

If the tube is thin enough, it will not modify area significantly, and we will still have

$$\|\Sigma_{s,t}\| \lesssim 2\|\Gamma_1\|,$$

for all $(s,t) \in [-1,1]^2$.

To be able to distinguish the min-max limit of the new sweepout from Γ_1, we need the former inequality to be strict. By Inequality 26, the value $\|\Sigma_{s,t}\|$ can equal $2\|\Gamma_1\|$ on the diagonal $s = t$ and near the origin $(0,0)$ of the parameter space $[-1,1]^2$. Hence, on the diagonal, we open the neck of $\Sigma_{s,t}$ given by the thin tube using the catenoid estimate, to produce a new 2-parameter sweepout $\{\Sigma'_{s,t}\}$, which satisfies

$$\|\Sigma_{s,t}\| < 2\|\Gamma_1\|, \tag{27}$$

for all $(s,t) \in [-1,1]^2$.

Define the min-max width associated with the family generated by this 2-parametric sweepout to be

$$\omega_2 := \inf_{\substack{\{\Gamma_{s,t}\}_{(s,t)\in\Delta} \\ \Gamma_{s,t}|_{\partial\Delta} = \Sigma_{s,t}}} \sup_{(s,t)\in\Delta} \|\Gamma_{s,t}\|.$$

By Inequality 27, we have $\omega_2 < 2\omega_1 = 2\|\Gamma_1\|$, so the new min-max sequence cannot converge to Γ_1 with multiplicity 2. On the other hand, if $\omega_2 = \omega_1$, by Lusternik-Schnirelman theory we have that there exist infinitely many minimal embedded 2-spheres. The idea behind this fact is that if there exist two points of equal min-max width, then we can apply a *universal pull-tight* such that any line between them produces a minimal 2-sphere. Finally, if

$$\omega_1 < \omega_2 < 2\omega_1,$$

the new min-max limit is a minimal 2-sphere distinct from Γ_1. □

Lecture 8

We now return to re-examining the ellipsoid shown in Example 4.2. We fix b, c, d and denote by $E(a) := E(a,b,c,d)$ the resulting ellipsoid. First, we know that there exists a planar minimal sphere $\Gamma_1 \subset E(a)$, given by $\Gamma_1 = E(a) \cap \{x_1 = 0\}$. Moreover, we observe that Γ_1 is **fixed** and independent of a. By the work of Haslhofer–Ketover (Theorem 4.6), we know that there exists a second minimal sphere, which we denote by $\Gamma_2(a) \subset E(a)$, such that

$$|\Gamma_2(a)| \le 2|\Gamma_1|.$$

as we showed in (27). In particular, the area bound of $\Gamma_2(a)$ does not depend on a.

As $a \to \infty$, it is clear to see that the ellipsoid $E(a)$ converges locally smoothly to a cylinder $\Gamma_1 \times \mathbb{R}$. Our next goal is to understand what happens to $\Gamma_2(a)$ in this limit.

Proposition 4.8 (Haslhofer - Ketover [HK19])**.** *As $a \to \infty$, the surface $\Gamma_2(a)$ converges as varifolds to $\{\Gamma_1 \times \{0\}\}$ with multiplicity 2.*

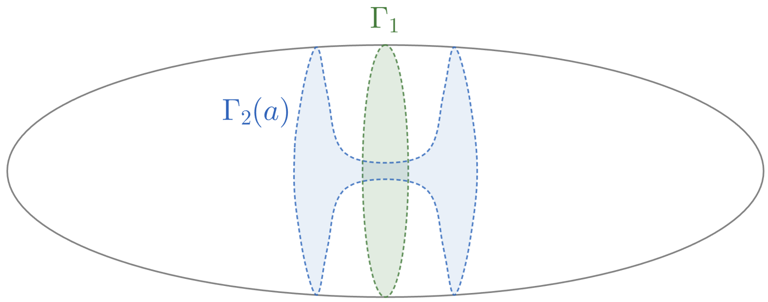

Figure 25: Surface convergence of $\Gamma_2(a)$ to $\{\Gamma_1 \times \{0\}\}$ with multiplicity 2

Proof. We start by noting that all the ellipsoids $E(a)$ have positive Ricci curvature. As we have seen, this implies that there does not exist any stable minimal surfaces in $E(a)$; also any 2 minimal surfaces in $E(a)$ must intersect due to Frankel's property. Since we have

$$|\Gamma_2(a)| \leqslant 2|\Gamma_1|,$$

it follows by Allard's compactness Theorem 2.33 that we may extract a stationary integral varifold $\Gamma_2(\infty)$ as a subsequential limit of the $\Gamma_2(a)$.

First, it is not hard to see that the only stationary integral varifolds in the limit manifold $\Gamma_1 \times \mathbb{R}$ are the slices of the form $\Gamma_1 \times \{a\}$, for some $a \in \mathbb{R}$. This follows from the maximum principle for varifolds (see Lemma B.1. in [CDL03] and its corrected proof in [CDL]), since any stationary integral varifold projecting to a (bounded) interval in the $\mathbb{R}-$factor can be compared on both sides by appropriate horizontal disks $\Gamma_1 \times \{a\}$, all of which are minimal.

On the other hand, the Frankel property implies that $\Gamma_2(a) \cap \Gamma_1 \neq \varnothing$ for all a. By using the monotonicity formula on a sequence of points $p_a \in \Gamma_1 \cap \Gamma_2(a)$, one can deduce that on a neighbourhood U of Γ_1, the area $(\Gamma_1 \cup \Gamma_2(a)) \cap U$ can be bounded from below as

$$\| (\Gamma_1 \cup \Gamma_2(a)) \cap U\| \geq \|\Gamma_1\| + C,$$

for some positive constant C independent of a. Consequently, in the limit as $a \to \infty$, the mass of $(\Gamma_1 \cup \Gamma_2(\infty)) \cap U$ is strictly larger than Γ_1, which implies that $\Gamma_2(\infty)$ must intersect U. Since U is an arbitrary neighbourhood of Γ_1, we deduce that $\Gamma_1 \subset \Gamma_2(\infty)$. Combining these observations, we deduce that $\Gamma_2(\infty)$ must be either a union of two slices $\Gamma_1 \times \{0\} + \Gamma_1 \times \{a\}$, for some $a \in \mathbb{R}$ or the single slice $\Gamma_1 \times \{0\}$ (with multiplicity one). Let us check that, in the first case, one actually has $a = 0$, and both slices coincide with $\Gamma_1 \times \{0\}$.

First, assume by contradiction that $a \neq 0$. Then, the Frankel property applied to each of the spaces $E(a)$ implies that the surfaces $\Gamma_2(a)$ must be connected. By using the monotonicity formula, one can check $\Gamma_2(a)$ cannot contain a small *neck* that might pinch off in the limit, forming the two slices. The idea is that otherwise, for a large enough we would have sub-Euclidean volume growth on the neck, contradicting the formula. One can also check that the only way for the surfaces to disconnect into two slices would be the pinching of such necks, so this situation cannot happen.

Finally, it remains to rule out the possibility that $\Gamma_2(\infty) = \Gamma_1 \times \{0\}$ with multiplicity one for the limit varifold. By Allard's regularity theorem, we know that if the convergence occurs with multiplicity 1, then it must happen smoothly (in C^∞_{loc}), in the sense of graphs. This fact is not direct, but can be obtained as a consequence of compactness results and a regularity theorem by Allard [LS84, Theorem 7.1]. Then, we would have $\Gamma_2(a) \xrightarrow{C^\infty_{\text{loc}}} \Gamma_1 \times \{0\}$ smoothly.

Applying the Frankel property again, we know that the $\Gamma_2(a)$ cannot lie on only one side of the limit $\Gamma_1 \times \{0\}$, but rather it must lie on both sides. This implies the existence of a Jacobi field φ on the limit surface Γ_1 which changes sign. Recall that on each $E(a)$, the Jacobi operator of a surface is given by

$$L_\Gamma^{(a)} = \Delta + |A|^2 + \mathsf{Ric}(\nu, \nu).$$

First, one can check that $\mathsf{Ric}(\nu,\nu) \to 0$ as $a \to \infty$, since $E(a)$ converges to $\mathbb{S}^2 \times \mathbb{R}$. The spheres $\Gamma_1 \times \{0\}$ also have $|A|^2 = 0$: they arise as the fixed points of an involution (the reflection map), hence they are totally geodesic. Thus, the Jacobi operator on the limit is $L_{\Gamma_1}^{E(a)} \to \Delta$ on Γ_1, the Laplacian. However, we obtain a contradiction, since there is no function $\varphi \in C^\infty(\Gamma_1)$ such that $\Delta\varphi = 0$ and φ has a sign change, as the only harmonic functions on the sphere $\mathbb{S}^2$ are constant.

Thus, the only option is that $\Gamma_2(a)$ converges in the limit to $2\Gamma_1 \times \{0\}$, as desired; this completes the proof. □

In this limit situation, the minimal surface $\Gamma_2(\infty)$ has multiplicity. Wang and Zhou have a related result in which they show examples of metrics in the 3-sphere such that any 2-parameter sweepout must be a minimal 2-sphere, which can be found in [WZ22]. This contrasts with the celebrated proof of the Multiplicity One Conjecture, proved by Xin Zhou. Roughly speaking, the conjecture states the following:

Theorem 4.9 (Multiplicity One Conjecture, Zhou [Zho20])**.** *Let (M, g) be a closed,orientable, three dimensional Riemannian manifold with a bumpy metric g. Then, the two-sided unstable minimal hypersurfaces obtained by min-max techniques have multiplicity one.*

With this result, we can sketch the proof of Theorem 4.7.

Proof of Theorem 4.7. Suppose first that there does not exist a stable minimal sphere in $(\mathbb{S}^3, g)$. Then if we apply Theorem 4.9 to a four parameter families of sweepouts of embedded spheres, we get at least four distinct embedded minimal spheres of multiplicity one. Now suppose that there does exist a stable minimal sphere, say $\mathbb{S}^2$. Now we separate $\mathbb{S}^3$ by this two sphere, obtaining two 3-balls B^3_1, B^3_2 with stable minimal boundary. Let us focus on one of these balls, which we denote by B^3. It is possible to *glue* the boundary of B^3 with a cylindrical end modeled by $\mathbb{S}^2 \times [0, \infty)$. This idea is due to Song [Son23]. Now, one can apply min-max theory in these non-compact manifolds, obtaining minimal spheres in B^3. In fact, by using a variant of Theorem 4.9, it is possible to construct at least two such spheres. Applying this procedure on the two balls, we obtain five minimal spheres: two for each ball B^3_i plus the stable minimal surface $\mathbb{S}^2$. □

5 The Smale conjecture

In the first four lectures of this course we saw that the existence of non-trivial sweepouts can be used to prove the existence of minimal surfaces, for example minimal 2-spheres in 3-spheres. Now we will use the *opposite* idea, that is, we will see that the non-existence of certain minimal surfaces implies the non-existence of trivial topology on certain spaces of surfaces. In this section we will see how this is used to prove the **Smale conjecture** for $\mathbb{RP}^3$.

In 1959, Smale proved that the inclusion of $\mathrm{Isom}(\mathbb{S}^2)$ in $\mathrm{Diff}(\mathbb{S}^2)$ is a homotopy equivalence, and conjectured the analogous result for $\mathbb{S}^3$, which was proved by Hatcher in 1982. In the case of $\mathbb{S}^3$, this theorem is equivalent to the following statements:

1. The space of maps $\mathsf{Emb}(\mathbb{S}^2, \mathbb{S}^3)$ retract onto $\mathbb{RP}^3$,
2. $\mathsf{Emb}(\mathbb{S}^2, \mathbb{S}^2 \times \mathbb{R})$ is contractible.

One can also ask for the generalized Smale conjecture;

Conjecture 5.1 (Generalized Smale conjecture)**.** *If (M, g) is a closed orientable Riemannian manifold of dimension 3, and g is a metric of constant sectional curvature ± 1, then the inclusion of Isom(M) in Diff(M) is a homotopy equivalence.*

Over the years many authors have proved the conjecture for special types of manifolds. Refer to [KL25] for a detailed discussion. One of the remaining cases, among others, was $\mathbb{RP}^3$. Considering the complexity of the Hatcher's proof, one of the long standing challenges in Yau's list of problems is to reprove the Conjecture for $\mathbb{S}^3$ or to prove the remaining case of Conjecture 5.1 using geometric analysis methods. In 2018 Bamler-Kleiner achieved both goals using Hamilton-Perelman's Ricci flow.

Recently, Ketover–Liokumovich gave a new proof of Conjecture 5.1 for the set of Lens spaces $L(p, q)$ (which includes the case of $\mathbb{RP}^3$ when $(p, q) = (2, 1)$) by using min-max methods. The first observation in the case of lens spaces is that we have the following equivalent formulation

Theorem 5.2 (Ketover–Liokumovich; Smale's conjecture for Lens Spaces)**.** *The space of Heegaard tori in round $L(p, q)$ retracts onto the subspace of Clifford tori.*

We will give a sketch for the case of $\mathbb{RP}^3 = L(2, 1)$ instead of general lens spaces. The general argument can be found in [KL25]. It's worth noting that the space of Clifford Tori in $\mathbb{RP}^3$ is homeomorphic to $\mathbb{RP}^2 \times \mathbb{RP}^2$.

Sketch of the proof of Theorem 5.2. Let $\mathcal{T}$ denote the space of Heegaard embeddings of tori in $\mathbb{RP}^3$; and $\mathcal{T}_C$, the space of Clifford tori in $\mathbb{RP}^3$. Since $\mathcal{T}$ has the homotopy type of CW-Complex, then by Whitehead's theorem it's enough to show that all relative homotopy groups $\pi_k(\mathcal{T}, \mathcal{T}_C)$ vanish. We will prove this fact by induction on k.

First, observe that we have a natural map, called the **Hurewicz map**, between the k-th homotopy group and the k-th homology group,

$$H : \pi_k(\mathcal{T}, \mathcal{T}_C) \to H_k(\mathcal{T}, \mathcal{T}_C),$$

which is defined as follows: let $\alpha \in \pi_k(\mathcal{T}, \mathcal{T}_C)$, so that $\alpha : I^k \to \mathcal{T}$ satisfies $\alpha|_{\partial I^k} \in \mathcal{T}_C$. Consider then the induced map $\alpha^* : H_k(I^k, \partial I^k) \to H_k(\mathcal{T}, \mathcal{T}_C)$, and let us define $H(\alpha) = \alpha^*(\beta)$, where β is the generator of $H_k(I^k, \partial I^k)$. We claim:

Claim 5.3. Given $k \geq 1$, and $a \in \pi_k(\mathcal{T}, \mathcal{T}_C)$, $H(a) = 0$.

Proof of the claim. Suppose the map $f' : I^k \to \mathcal{T}$, such that $f'|_{\partial I^k} \subset \mathcal{T}_C$, represents the homotopically non-trivial map class of a. It can be shown that $p : (\mathrm{Diff}(\mathbb{RP}^3), \mathrm{Isom}(\mathbb{RP}^3)) \to (\mathcal{T}, \mathcal{T}_C)$ is a homotopy equivalence of pairs. Let q be the homotopy inverse, and define $f := p \circ q \circ f'$. It is easy to see that f and f' are homotopic. Thus it is enough to show that $H(f)$ is trivial. Using the family of diffeomorphisms $b = q(f')$, it is possible extend f' and get a one parameter family of embeddings $\hat{f} : I^k \times [-1, 1] \to \mathbb{RP}^3$, then define the sweepouts as $\tilde{f}(x, t) = \hat{f}(x, t)(\mathbb{T}^2)$ such that $\{\tilde{f}(x, t)\}_{t\in[-1,1]}$ is a Heegaard foliation of $\mathbb{RP}^3$. This can be viewed as a way to "extend" our family of tori onto both sides. Now we define the space of sweepouts preserving the boundary data of the extension $\tilde{f}$, i.e.,

$$\Pi = \Big\{\text{all sweepouts } \ \Gamma_{x,t} \text{ agreeing with } \ \tilde{f} \text{ at } \ \partial(\mathbb{D}^k \times [-1, 1])\Big\}.$$

This is a $(k + 1)-$parameter min-max family: the parameter space is this cylinder. Let us consider the associated min-max width

$$\omega = \inf_{\Gamma_{x,t}\in\Pi} \sup |\Gamma_{x,t}|. \tag{28}$$

We now prove the min-max width must be $\omega = \pi^2$. One can see that in the sweepouts appear minimal tori which are exactly the projection of some Clifford torus in $\mathbb{S}^3$, so $\omega \geq \pi^2$. Now assume by contradiction that $\omega > \pi^2$. Then there exists a minimizing sequence of sweepouts and the corresponding min-max sequence converges to a minimal surface possibly with multiplicity. Using the genus bound we already observed the resulting surface cannot increase the genus, thus, the possible limits are a torus or an $\mathbb{RP}^2$, possibly with some multiplicity k. The multiplicity-one result of Zhou rules out the possibility of a multiplicity$-k$ projective plane, as its double cover is unstable in $\mathbb{S}^3$. Hence, the limit must be a torus; by Brendle's proof of the Lawson conjecture, we know that any minimal torus must be the Clifford torus. However, this has area π^2, giving a contradiction. Combining these results, we deduce the existence of a minimizing sequence $\Gamma^i_{x,t}$ such that

$$\max |\Gamma^i_{x,t}| \to \pi^2.$$

Accordingly, we obtain a sequence of manifolds $M_i \subset \mathbb{D}^k \times [-1, 1]$, where each $\partial M_i = \partial \mathbb{D}^k \times \{0\}$. By the construction of $\Gamma^i_{x,t}$, one can see that each family $\{\Gamma^i(x, 0)\}_{x\in I^k}$ is homotopic to $\{\tilde{f}(x, 0)\}_{x\in I^k}$, hence by the definition of Hurewicz map it follows that $H(a) = [(\Gamma^i)^*\beta]$, where β is the generator of $H_k(I^k \times [-1, 1], \partial I^k \times \{0\})$. Now to prove $[(\Gamma^i)^*\beta] = [0]$, we use a Lusternik-Schnirelmann argument to obtain a manifold in the parameter space, joining the equatorial points, such that each point of the manifold is close to a Clifford torus. Then we use Hatcher's argument to retract the "hair" of this family so that each element contains in a tubular neighbourhood. This argument can be summarized as follows: we previously had an unknown family of manifolds, and we modify it into a family that is close to Clifford tori and contains $\mathbb{T}^2 \times [-r, r]$ in it. Then we finish the proof by using arguments of Hatcher-Ivanov's work on Haken manifolds. $\square$

Claim 5.4. $\pi_1(\mathcal{T}, \mathcal{T}_C) = 0$

We assume the first induction step 5.4 and prove the rest of the induction argument. Assume $\pi_i(\mathcal{T}, \mathcal{T}_C) = 0$ for all $i < k$, then we have the relative Hurewicz theorem which states the map

$$H' : \pi_k(\mathcal{T}, \mathcal{T}_C)/\pi_1(\mathcal{T}_C) \to H_k(\mathcal{T}, \mathcal{T}_C) \tag{29}$$

is a bijection. But recall we already proved that the map H is trivial, thus we get $H' = 0$, but then H' is a bijection which implies $\pi_k(\mathcal{T}, \mathcal{T}_C) = \pi_1(\mathcal{T}_C)$. However, the induction step for $k = 1$ implies $\pi_1(\mathcal{T}_C) = 0$, which completes the proof. $\square$

Remark 5.5. This result is more difficult in $\mathbb{S}^3$: in the above argument, the key step was obtaining the extension of f to one parameter sweepouts. This construction relies on the Smale conjecture for $\mathbb{S}^3$. In particular the existence of the homotopy equivalence map is assured by the Smale conjecture of $\mathbb{S}^3$.

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