

EXACT SOLUTION OF THE TWO-MODE MODEL OF MULTICOMPONENT BOSE-EINSTEIN CONDENSATES.

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(Communicated by Bjorn Sandstede)

ABSTRACT. We find the explicit solution of the so-called two-mode model for multicomponent Bose-Einstein condensates. We prove that all the solutions are constants or periodic functions and give explicit formulae for them.

1. Introduction. One of the most remarkable achievements of contemporary physics has been the realization of Bose-Einstein condensation with ultracold atomic gases [1]. In the regime of very low temperatures, the gas dynamics is modelled by the so called Gross-Pitaevskii equation, which reads

$$i\hbar \frac{\partial}{\partial t} \Psi = \left[-\frac{\hbar^2}{2m} \Delta + \frac{1}{2} m\omega^2 \mathbf{r}^2 + U|\Psi|^2 \right] \Psi, \quad (1)$$

where m is the mass of the atoms, \hbar is the Planck constant, U is a parameter related to the probability of collisions between individual atoms and $\Delta = \sum_{j=1}^D \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator in D dimensions. The quantity $|\Psi(t, \mathbf{r})|^2$ has the physical meaning of the local density of the gas and the total number of atoms, which is given by $N := \|\Psi\|_{L^2}^2 = \int d^D \mathbf{r} |\Psi(t, \mathbf{r})|^2$, is conserved under time evolution. The potential $V(\mathbf{r}) = \frac{1}{2} m\omega^2 \mathbf{r}^2$, where $\mathbf{r}^2 = \sum_{j=1}^D r_j^2$ and $\omega \in \mathbb{R}^+$, is an external potential generated by magnetic fields. The presence of this potential is responsible for the physical confinement of the atoms which would otherwise spread out due to the combined effect of dispersion (term with Laplacian) and the nonlinear interaction (which is repulsive).

These type of systems are naturally three-dimensional, however, in many situations in which the atoms are tightly confined along one of the dimensions, it has been shown that a two-dimensional description of the problem is possible. The

1991 *Mathematics Subject Classification.* 74J30, 34A05, 35Q55, 37N20.

Key words and phrases. Nonlinear waves, Exactly solvable models, Bose-Einstein condensation.

only change is that the scattering parameter U and number of particles N are then understood as effective parameters proportional to the real ones [2].

Although our analysis will be related to dynamical phenomena in Bose-Einstein condensates equation (1) is an equation of nonlinear Schrödinger type and thus our results to be described later will have a much larger range of applicability because of the well-known universality of this model equation [3].

In this paper we concentrate on the analysis of multicomponent systems. When two different type of atoms of the same mass (usually two different hyperfine levels of the same atomic specie), which occupy the same spatial region are simultaneously cooled down below the critical temperature, each type of atom is described by a single function $\Psi_1(\mathbf{r}, t)$, $\Psi_2(\mathbf{r}, t)$ so that the system is described by the equations

$$i\hbar \frac{\partial}{\partial t} \Psi_j = \left[-\frac{\hbar^2}{2m} \Delta + \frac{1}{2} m \omega^2 \mathbf{r}^2 + \sum_{k=1}^2 U_{jk} |\Psi_k|^2 \right] \Psi_j, \quad j = 1, 2 \quad (2)$$

and the number of particles of the j th type is given by $N_j = \int d^D \mathbf{r} |\Psi_j(t, \mathbf{r})|^2$. These equations are a generalization of Eq. (1) to the case of two atomic species. Here U_{jk} are constants controlling the nonlinear behavior proportional to the mutual collision amplitudes and $U_{21} = U_{12}$.

It is convenient to work with a set of adimensional units. To this end we scale coordinates and time by $\mathbf{r} \rightarrow r_0 \mathbf{r}, t \rightarrow t/\omega$ with $r_0 = \sqrt{\hbar/m\omega}$, (hereafter \mathbf{r} and t stand for dimensionless coordinates and time) and introduce new functions ψ_j by $\Psi_j = \psi_j / r_0^{D/2}$ which are normalized as $\int d^D \mathbf{r} |\psi_j(t, \mathbf{r})|^2 = N_j$. Eqs. (2) read now

$$i \frac{\partial}{\partial t} \psi_1 = \left[-\frac{1}{2} \Delta + \frac{1}{2} r^2 + u_{11} |\psi_1|^2 + u_{12} |\psi_2|^2 \right] \psi_1, \quad (3a)$$

$$i \frac{\partial}{\partial t} \psi_2 = \left[-\frac{1}{2} \Delta + \frac{1}{2} r^2 + u_{21} |\psi_1|^2 + u_{22} |\psi_2|^2 \right] \psi_2 \quad (3b)$$

with $u_{jk} = U_{jk} r_0^D / \hbar \omega$.

These type of physical systems were first generated experimentally in 1997 [4]. One of the most remarkable achievements using these systems was the first experimental generation of vortices in Bose-Einstein condensates reported in Ref. [6].

Let us now present the concept of solitary wave or stationary solution of Eq. (1) which in adimensional units can be written as

$$i \frac{\partial}{\partial t} \psi = \left[-\frac{1}{2} \Delta + \frac{1}{2} r^2 + u |\psi|^2 \right] \psi. \quad (4)$$

By definition *solitary waves* are solutions of the form $\psi(t, \mathbf{r}) = \phi_\mu(\mathbf{r}) e^{-i\mu t}$ where the pair $\mu \in \mathbb{R}$ and $\phi_\mu(\mathbf{r})$ satisfy the elliptic problem

$$\mu \phi_\mu = \left[-\frac{1}{2} \Delta + \frac{1}{2} r^2 + u |\phi_\mu|^2 \right] \phi_\mu, \quad (5)$$

with boundary conditions $\lim_{\mathbf{r} \rightarrow \infty} \phi_\mu(\mathbf{r}) = 0$. These solutions are stationary points of the functional

$$E(\phi) = \frac{1}{2} \int d^D \mathbf{r} \left[|\nabla \phi|^2 + r^2 |\phi|^2 + u |\phi|^4 \right] \quad (6)$$

of fixed L^2 -norm. Of the full family of solutions $\phi_\mu(\mathbf{r})$ of Eq. (5) the one which corresponds to the global minimum of the functional (6) is called the ground state and we will denote it by $\phi_1(\mathbf{r})$.

In the linear case ($U = 0$) this system has a numerable set of solutions. Say, in the $D = 2$ case this set can be chosen as $\phi_{kl}(\mathbf{r}) = H_k(x)H_l(y)e^{-(x^2+y^2)/2}$, where $\mu_{kl} = k + l + 1$, $k, l \in \mathbb{N}$ and $H_{k,l}$ are the Hermite polynomials. For the nonlinear repulsive case ($U > 0$) it is has been proven in limited situations [7] and conjectured for the general case that the situation is the same, i.e., for each value of the norm N there is a numerable set of solutions $\{\mu_n, \phi_n\}$.

It is known that the ground state of Eq. (5) is a unique positive radial solution [8] which is finite when $\mathbf{r} \rightarrow 0$. General properties of the solutions of Eq. (5) have been studied in a more general context in [9, 10].

When the nonlinear parameter is large enough [17] it has been checked numerically that the first excited state, $\phi_2(\mathbf{r})$, is a solution of the so-called vortex type. The simplest vortex solutions of Eq. (5) in the two-dimensional case is of the form $\phi_2(\mathbf{r}) = R_2(r)e^{i\theta}$, where $\mu_2 \in \mathbb{R}$, $R_2(r) : \mathbb{R}^+ \rightarrow \mathbb{R}$ and (r, θ) are the polar radius and angle. The solutions $\mu_2, R_2(r)$ are to be determined from the equations

$$\mu_2 R_2 = -\frac{1}{2} \left(\frac{d^2 R_2}{dr^2} + \frac{1}{r} \frac{dR_2}{dr} + \frac{1}{r^2} R_2 \right) + \frac{1}{2} r^2 R_2 + u R_2^3 \quad (7a)$$

$$2\pi \int r R_2(r) dr = N \quad (7b)$$

It is possible to prove that this equation has regular solutions satisfying the condition that $R_2(r) \sim r$ when $r \rightarrow 0$ and $R_2(r) \leq r e^{-r^2}$, $r \rightarrow \infty$. The name *vortex* comes from the fact that if the Bose-Einstein condensate is a superfluid then the gradient of the nonhomogeneous phase term is physically associated to a closed circulation of the quantum fluid (see e.g. [11] for details). If $\Theta = \arg \psi$ the term *vortex* refers to solutions for which $m = \frac{1}{2\pi} \oint \nabla \Theta d\ell$ is an integer nonzero value for any closed curve γ containing a point – the so called *vortex core* – (x_0, y_0) where $\psi(x_0, y_0) = 0$.

There has been a great interest on multicomponent systems in which one of the atomic species has initially a vortex, i.e., the corresponding $\psi_j(\mathbf{r}, 0)$ is of vortex type and $\psi_{\bar{j}}(\mathbf{r}, 0)$ with $\bar{j} \neq j$ is a nodeless function. This interest was triggered by the experimental results reported in Ref.[6] where one of the possible configurations with $j = 1, 2$ was found to be stable and the other one unstable. In recent works [12, 13] numerical simulations of Eqs. (3) have been used to show that the dynamics of the unstable configuration can be understood within the framework of mean field theories for the double-condensate system. Later [14], it was shown that a simple model was able to describe accurately the dynamics of the multiple-condensate system in a particular two dimensional situation and an specific parameter region close to the experimental parameters of Ref. [6].

In this paper we concentrate on the analysis of the simple model reported in Ref. [14] proving that the system is integrable and that all the solutions are either periodic or constants (equilibria). This proves that the tendency of the system to exhibit periodic exchange of vortices between both atomic species shown in Refs.[12, 13, 14] is not the consequence of a fortunate choice of parameters. We also study the period of the solutions as a function of the parameters of the problem and compare quantitatively the analytical solutions with numerical simulations.

2. The two-mode model in two dimensions. We will derive here the two-mode model following a different formalism than that of Ref. [14]. The idea of the

method is to use the reduced Lagrangian approach. Equations (3) can be obtained from the variational problem

$$\frac{\delta}{\delta \bar{\psi}_n} \int dt L(t) = 0 \quad (8)$$

where

$$L(t) = \int d^2 \mathbf{r} \mathcal{L}(t, \mathbf{r}) \quad (9)$$

(hereafter we restrict ourselves to the two-dimensional case $D = 2$) with

$$\begin{aligned} \mathcal{L}(t, \mathbf{r}) = & \sum_n \left\{ -i \bar{\psi}_n \frac{\partial \psi_n}{\partial t} + \frac{1}{2} (\nabla \bar{\psi}_n, \nabla \psi_n) + \frac{1}{2} \mathbf{r}^2 |\psi_n|^2 \right\} \\ & + \frac{1}{2} \sum_{mn} u_{mn} |\psi_m|^2 |\psi_n|^2 \end{aligned} \quad (10)$$

In terms of some basis,

$$\psi_n(t, \mathbf{r}) = \sum_k a_{nk}(t) \phi_k(\mathbf{r}), \quad (11)$$

associated with the linear problem

$$\left[-\frac{1}{2} \Delta + \frac{1}{2} r^2 \right] \phi_j(\mathbf{r}) = E_j \phi_j(\mathbf{r}), \quad (12)$$

$$\int d^2 \mathbf{r} \bar{\phi}_j(\mathbf{r}) \phi_k(\mathbf{r}) = \delta_{jk} \quad (13)$$

the Lagrangian can be rewritten as follows:

$$\begin{aligned} L(t) = & \sum_{nj} \left\{ -i \bar{a}_{nj} \frac{\partial a_{nj}}{\partial t} + E_j |a_{nj}|^2 \right\} \\ & + \sum_{mn} u_{mn} \sum_{jkj'k'} \bar{a}_{mj} \bar{a}_{nk} a_{mj'} a_{nk'} \int d^2 \mathbf{r} \bar{\phi}_j \bar{\phi}_k \phi_{j'} \phi_{k'} \end{aligned} \quad (14)$$

In what follows we will use a limited expansion, i.e. we keep in (11) only two modes:

$$\psi_n(t, \mathbf{r}) = \sum_{k=1}^2 a_{nk}(t) \phi_k(\mathbf{r}), \quad (15)$$

where ϕ_1 is the mode resembling the ground state and ϕ_2 corresponds to a state with a vortex. Is this approximation accurate enough to describe the dynamics contained in Eqs. (3)? It is evident that in a general situation the answer is negative but in this paper our interest is in the dynamics of special configurations having initially a vortex in any of the components plus ground state in the other one. Thus, our choice for is the simplest choice which is able to approximate the initial data of interest for our physical situation. Moreover, numerical stability analysis of the full partial differential equations ruling the phenomena [12] shows that when vortex configurations are unstable the destabilization is caused by the growth of gaussian (ground-state) modes. This numerical evidence makes us expect

that our choice will be able to describe at least the stationary states and the first stages of the unstable cases.

Finally, the fact that the dynamics obtained from the dynamics of the full PDE system has some sort of recurrent dynamics leads us to the hope that the phenomena could more accurately be described by this crude approximation than one could expect in principle. In fact, despite its simplicity and crudeness this approximation has been shown to retain many qualitative features of the dynamics of the system [14]. Further comparisons with the numerical results obtained from simulations of the full problem (3) to be presented later in this paper will also support our statement.

In this paper we make a more detailed derivation and mathematical analysis of this model together with a deeper physical analysis of its range of validity and implications.

Inserting (15) into Eq. (14) and noting that ϕ_1 is axially symmetric while $\phi_2 \propto \exp(i\theta)$ one can obtain that of all 16 integrals in the nonlinear part of the Lagrangian only 6 are nonzero, namely the ones which correspond to the following values of the indices: $(j, k, j', k') = (1, 1, 1, 1), (1, 2, 1, 2), (1, 2, 2, 1), (2, 1, 1, 2), (2, 1, 2, 1), (2, 2, 2, 2)$. Thus the truncated Lagrangian becomes

$$L(t) = \sum_{n,j=1}^2 \left\{ -i\bar{a}_{nj} \frac{\partial a_{nj}}{\partial t} + E_j |a_{nj}|^2 \right\} + \sum_{mn} u_{mn} \left\{ \sum_{jk} \frac{1}{2} C_{jk} |a_{mj}|^2 |a_{nk}|^2 + C_{12} \bar{a}_{m1} a_{m2} \bar{a}_{n2} a_{n1} \right\} \quad (16)$$

where $C_{jk} = \int d^2\mathbf{r} |\phi_j|^2 |\phi_k|^2$.

Namely this Lagrangian we consider as describing the two-mode model in two dimensions. As discussed before, this model allows a very complete description of the dynamics of the system and will be the starting point for our discussion. In particular we will try to understand why most of the simulations of the two-mode model and the full PDE system (3) lead frequently to periodic solutions. Specifically our main result, apart from obtaining the explicit form of the solutions, will be the proof that all solutions of this system are periodic functions of time and the analysis of the dependence of this period on the relevant parameters of the problem.

3. Analysis of the two-mode model.

3.1. Reformulation of the model. Varying Lagrangian (17) with respect to \bar{a}_{nj} , $n, j = 1, 2$ we get the evolution equations for the coefficients a_{nj} :

$$i \frac{d}{dt} a_{nj} = E_j a_{nj} + \left(\sum_{m,k=1}^2 u_{nm} C_{jk} |a_{mk}|^2 \right) a_{nj} + C_{12} u_{nn} |a_{n\bar{j}}|^2 a_{nj} + C_{12} u_{12} \bar{a}_{\bar{n}\bar{j}} a_{\bar{n}j} a_{n\bar{j}} \quad (17)$$

where \bar{n} and \bar{j} are the complementary values of the indices n and j : $\bar{n} = 3 - n$, $\bar{j} = 3 - j$. Taking the real and imaginary parts of these four equations for four complex variables one can get eight equations for moduli ρ_{nj} and phases θ_{nj} ,

$$a_{nj} = \rho_{nj} e^{i\theta_{nj}} \quad (18)$$

The equations for moduli can be written as

$$d\rho_{11}^2/dt = -2C_{12}u_{12} \rho_{11}\rho_{12}\rho_{21}\rho_{22} \sin \Phi, \quad (19a)$$

$$d\rho_{12}^2/dt = +2C_{12}u_{12} \rho_{11}\rho_{12}\rho_{21}\rho_{22} \sin \Phi, \quad (19b)$$

$$d\rho_{21}^2/dt = +2C_{12}u_{21} \rho_{11}\rho_{12}\rho_{21}\rho_{22} \sin \Phi, \quad (19c)$$

$$d\rho_{22}^2/dt = -2C_{12}u_{21} \rho_{11}\rho_{12}\rho_{21}\rho_{22} \sin \Phi, \quad (19d)$$

where

$$\Phi = \theta_{11} - \theta_{12} - \theta_{21} + \theta = \arg \frac{a_{11}a_{22}}{a_{12}a_{21}} \quad (20)$$

while the equations for phases are

$$0 = \frac{d\theta_{nj}}{dt} + E_j + \sum_{m,k=1}^2 u_{nm} C_{jk} \rho_{mk}^2 \quad (21)$$

$$+ C_{12} u_{nm} \rho_{n\bar{j}}^2 + C_{12} u_{12} \frac{\rho_{11}\rho_{12}\rho_{21}\rho_{22}}{\rho_{n\bar{j}}^2} \cos \Phi.$$

These equations must be complemented with the appropriate initial conditions for ρ_{kl} and Φ .

$$\rho_{nj}(t_0) = \hat{\rho}_{nj} \quad \theta_{nj}(t_0) = \hat{\theta}_{nj} \quad n, j = 1, 2 \quad (22)$$

The form of the two-mode equations [Eqs. (19), (21)] presented here is slightly different from that of Ref. [14], but is more suitable for our purposes in this paper.

3.2. Solution of the equations. Let us first note that, as follows from (19), the two-mode model equations have three constants of motion which are quadratic in ρ . Two of them are closely related to the normalization conditions for the functions ψ_n ,

$$\rho_{11}^2 + \rho_{12}^2 = \text{const} = N_1 \quad (23)$$

$$\rho_{21}^2 + \rho_{22}^2 = \text{const} = N_2 \quad (24)$$

while the third one can be chosen as

$$\rho_{12}^2 + \rho_{22}^2 = \text{const} = M \quad (25)$$

This means that of the four quantities ρ_{nj}^2 only one is essentially independent. For the following purposes it is convenient to introduce Γ ,

$$\Gamma = \sum_{nj} \gamma_{nj} \rho_{nj}^2, \quad (26)$$

with

$$\gamma_{11} = C_{11}(u_{11} - u_{12}) + C_{12}(u_{12} - 2u_{11}), \quad (27)$$

$$\gamma_{12} = C_{22}(u_{12} - u_{11}) + C_{12}(2u_{11} - u_{12}), \quad (28)$$

$$\gamma_{22} = C_{22}(u_{22} - u_{12}) + C_{12}(u_{12} - 2u_{22}), \quad (29)$$

$$\gamma_{21} = C_{11}(u_{12} - u_{22}) + C_{12}(2u_{22} - u_{12}). \quad (30)$$

which satisfies

$$\frac{d\Gamma}{dt} = 2\gamma_* C_{12} \rho_{11}\rho_{12}\rho_{21}\rho_{22} \sin \Phi, \quad (31)$$

where

$$\gamma_* = u_{12} (\gamma_{12} + \gamma_{21} - \gamma_{11} - \gamma_{22}). \quad (32)$$

and the initial data are $\hat{\Gamma} \equiv \Gamma(0) = \sum_{nj} \gamma_{nj} \hat{\rho}_{nj}^2$.

The following relations are then evident:

$$\rho_{11}^2 = \hat{\rho}_{11}^2 - \frac{u_{12}}{\gamma_*} (\Gamma - \hat{\Gamma}) \quad (33a)$$

$$\rho_{12}^2 = \hat{\rho}_{12}^2 + \frac{u_{12}}{\gamma_*} (\Gamma - \hat{\Gamma}), \quad (33b)$$

$$\rho_{21}^2 = \hat{\rho}_{21}^2 + \frac{u_{12}}{\gamma_*} (\Gamma - \hat{\Gamma}), \quad (33c)$$

$$\rho_{22}^2 = \hat{\rho}_{22}^2 - \frac{u_{12}}{\gamma_*} (\Gamma - \hat{\Gamma}), \quad (33d)$$

with $\hat{\Gamma} = \sum_{nk} \gamma_{nk} \hat{\rho}_{nk}^2$. These equations imply that all the relevant densities may be obtained from a single quantity, $\Gamma(t)$, and our goal now will be to find an equation ruling its dynamics. Inserting representation (33) into Eqs. (21) and summarizing one can get

$$0 = \frac{d\Phi}{dt} + \Gamma + C_{12} u_{12} \rho_{11} \rho_{12} \rho_{21} \rho_{22} \left(\frac{1}{\rho_{11}^2} - \frac{1}{\rho_{12}^2} - \frac{1}{\rho_{21}^2} + \frac{1}{\rho_{22}^2} \right) \cos \Phi \quad (34)$$

From these equations we now deduce one equation for Γ . Let us notice that Eq. (34) can be rewritten as

$$\frac{d}{dt} \rho_{11} \rho_{12} \rho_{21} \rho_{22} \cos \Phi = \frac{1}{2C_{12}\gamma_*} \Gamma \frac{d\Gamma}{dt} \quad (35)$$

and solved:

$$\rho_{11} \rho_{12} \rho_{21} \rho_{22} \cos \Phi - \hat{\rho}_{11} \hat{\rho}_{12} \hat{\rho}_{21} \hat{\rho}_{22} \cos \hat{\Phi} = \frac{1}{4C_{12}\gamma_*} (\Gamma^2 - \hat{\Gamma}^2) \quad (36)$$

where $\hat{\Phi}$ is the initial value of Φ :

$$\hat{\Phi} = \hat{\theta}_{11} - \hat{\theta}_{12} - \hat{\theta}_{21} + \hat{\theta}_{22} \quad (37)$$

Using the last equation and (33), Eq. (31) for $d\Gamma/dt$ can be presented as

$$\left(\frac{d\Gamma}{dt} \right)^2 = \mathcal{P}_4(\Gamma) \quad (38)$$

where

$$\mathcal{P}_4(\Gamma) = -\frac{1}{4} \left[\Gamma^2 - \hat{\Gamma}^2 + 4\gamma_* C_{12} \hat{\rho}_{11} \hat{\rho}_{12} \hat{\rho}_{21} \hat{\rho}_{22} \cos \hat{\Phi} \right]^2 + \frac{4C_{12}^2 u_{12}^4}{\gamma_*^2} (\Gamma - \Gamma_{11}) (\Gamma - \Gamma_{12}) (\Gamma - \Gamma_{21}) (\Gamma - \Gamma_{22}) \quad (39)$$

and the constants Γ_{nk} , $n, k = 1, 2$ are defined as $\Gamma_{11} = \hat{\Gamma} + \gamma_* \hat{\rho}_{11}^2 / u_{12}$, $\Gamma_{12} = \hat{\Gamma} - \gamma_* \hat{\rho}_{12}^2 / u_{12}$, $\Gamma_{21} = \hat{\Gamma} - \gamma_* \hat{\rho}_{21}^2 / u_{12}$, $\Gamma_{22} = \hat{\Gamma} + \gamma_* \hat{\rho}_{22}^2 / u_{12}$. Thus, the solution of our equations can be obtained from a single equation for Γ , which, using the previous

equations, allows to obtain the expressions for $\rho_{jk}(t)$. In particular Eq. (38) leads to

$$t - t_0 = \int_{\hat{\Gamma}}^{\Gamma(t)} \frac{d\Gamma'}{\sqrt{\mathcal{P}_4(\Gamma')}}. \quad (40)$$

Let us rewrite the polynomial (39) in terms of its roots as follows

$$\mathcal{P}_4(\Gamma) = \mathcal{P}_* (\Gamma - P_1) (\Gamma - P_2) (\Gamma - Q_1) (\Gamma - Q_2) \quad (41)$$

where $\mathcal{P}_* = 4C_{12}^2 u_{12}^4 / \gamma_*^2 - 1/4$ and P_1, P_2, Q_1, Q_2 are the roots of \mathcal{P}_4 . There are at least two real roots P_1, P_2 of \mathcal{P}_4 satisfying that $\hat{\Gamma}$ is either a root or is located between two of them, i.e. $P_1 < \hat{\Gamma} < P_2$.

To prove the last affirmation let us first compute

$$\mathcal{P}_4(\hat{\Gamma}) = 4C_{12}^2 \gamma_*^2 \hat{\rho}_{11} \hat{\rho}_{12} \hat{\rho}_{21} \hat{\rho}_{22} \sin^2 \hat{\Phi} > 0.$$

Note that if the initial data are such that if $\hat{\Phi} = 0$ then $\hat{\Gamma}$ is a real root of $\mathcal{P}_4(\Gamma)$.

To understand what is the behavior when $\Phi \neq 0$ we evaluate

$$\begin{aligned} \mathcal{P}_4(\Gamma_{11}) &= -\frac{1}{4} \left[\Gamma_{11}^2 - \hat{\Gamma}^2 + 4\gamma_* C_{12} \hat{\rho}_{11} \hat{\rho}_{12} \hat{\rho}_{21} \hat{\rho}_{22} \cos \hat{\Phi} \right]^2 < 0, \\ \mathcal{P}_4(\Gamma_{12}) &= -\frac{1}{4} \left[\Gamma_{12}^2 - \hat{\Gamma}^2 + 4\gamma_* C_{12} \hat{\rho}_{11} \hat{\rho}_{12} \hat{\rho}_{21} \hat{\rho}_{22} \cos \hat{\Phi} \right]^2 < 0. \end{aligned}$$

This means that there is at least one real root located in each of the intervals $P_1 \in [\Gamma_{11}, \hat{\Gamma}]$ and $P_2 \in [\hat{\Gamma}, \Gamma_{12}]$.

As a corollary we get that $\mathcal{P}_4(\Gamma) > 0$ for $P_1 < \Gamma < P_2$. Concerning the other two roots $Q_{1,2}$ ($Q_2 > Q_1$ if real), nothing may be said in general. However, for the case $\mathcal{P}_* > 0$ it is clear that $\mathcal{P}_4(\pm\infty) > 0$ and then $Q_{1,2}$ are real numbers. When $\mathcal{P}_* < 0$ it happens that Q_1, Q_2 could be (and, in fact, they are) complex conjugate roots for certain parameter ranges. In any case, the boundedness of $\Gamma \in [P_1, P_2]$ ensures the periodicity of the solutions.

Let us now proceed to find an explicit form for the solutions. First, to present polynomial $\mathcal{P}_4(\Gamma)$ in the canonical form we make the transform:

$$\Gamma(t) = \frac{\alpha Y + \beta}{Y + \delta}, \quad (42)$$

where

$$\alpha = \frac{1}{2} (\delta + 1) P_2 - \frac{1}{2} (\delta - 1) P_1, \quad (43)$$

$$\beta = \frac{1}{2} (\delta + 1) P_2 + \frac{1}{2} (\delta - 1) P_1, \quad (44)$$

$$\delta = \frac{\sqrt{|P_1 - Q_1| |P_1 - Q_2|} + \sqrt{|P_2 - Q_1| |P_2 - Q_2|}}{\sqrt{|P_1 - Q_1| |P_1 - Q_2|} - \sqrt{|P_2 - Q_1| |P_2 - Q_2|}}. \quad (45)$$

Equivalently we may also write

$$\Gamma(t) = P_0 + \frac{\Delta}{\delta} \frac{Y}{\delta + Y} \quad (46)$$

with the quantities Δ and P_0 being given by

$$\Delta = \alpha\delta - \beta = (\delta^2 - 1) (P_2 - P_1), \quad (47)$$

$$P_0 = \frac{\beta}{\delta} = \frac{1}{2\delta} [(\delta + 1) P_2 + (\delta - 1) P_1], \quad (48)$$

It is clear that $P_1 < P_0 < P_2$ and thus $\mathcal{P}(P_0) > 0$. In terms of Y the polynomial $\mathcal{P}_4(\Gamma)$ can be written as

$$\mathcal{P}(\Gamma) = \delta^4 \mathcal{P}(P_0) \frac{(1 - Y^2)(1 - k^2 Y^2)}{(\delta + Y)^4}. \quad (49)$$

where

$$k = \frac{\sqrt{|Q_2 - P_1||P_2 - Q_1|} - \epsilon \sqrt{|P_1 - Q_1||Q_2 - P_2|}}{\sqrt{|Q_2 - P_1||P_2 - Q_1|} + \epsilon \sqrt{|P_1 - Q_1||Q_2 - P_2|}}. \quad (50)$$

with

$$\epsilon = \begin{cases} 1 & \text{Im} Q_j = 0 \\ -1 & Q_1 = \bar{Q}_2 \end{cases} \quad (51)$$

Thus, the function $Y(t)$ satisfies the equation

$$\frac{dY}{dt} = \Omega \sqrt{(1 - Y^2)(1 - k^2 Y^2)} \quad (52)$$

with

$$\Omega = \frac{\delta^2}{\Delta} \sqrt{\mathcal{P}(P_0)} \quad (53)$$

Equation (52) can be solved as follows:

$$Y(t) = \text{sn}(\Omega(t - \hat{t}), k), \quad (54)$$

where the constant \hat{t} should be determined from the initial conditions by solving the equation

$$\text{sn}(\Omega(t_0 - \hat{t}), k) = \frac{\Delta}{\alpha - \hat{\Gamma}} - \delta. \quad (55)$$

This allows us to get an explicit solution for our problem. To do so let us first use Eq. (54) to find

$$\Gamma(t) = \frac{\alpha \text{sn}(\Omega(t - \hat{t}), k) + \beta}{\text{sn}(\Omega(t - \hat{t}), k) + \delta}. \quad (56)$$

Thus, inserting this explicit expression for $\Gamma(t)$ and Eqs. (33) one may get explicit expressions for $\rho_{nj}(t)$, $n, j = 1, 2$.

The function sn is periodic, and then the period of our solutions is

$$T = \frac{4K(k)}{\Omega} \quad (57)$$

where the quantity $K(k) = \int_0^{\pi/2} d\theta / \sqrt{1 - k^2 \sin^2 \theta}$ is the complete elliptic integral of first kind [16].

4. Applications and discussion.

4.1. Comparison of the two-mode model with the results from the PDEs.

The explicit forms of the solutions given by (56) together with (33) and the formula for the period (57) are the main results of this paper. Although we tried to justify *a priori* the particular choice that we have used to model the initial problem, a more detailed comparison is in order.

In general, we must say that for the parameter ranges which we have explored in more detail the agreement of the two-mode model with the simulations of the partial differential equations (3) is good. For instance, in Fig. 1 it is shown the period of the vortex transfer as a function of the initial conditions $\rho_{jk}(0)$ for a specific choice of the parameters with a very good result for the two-mode model.

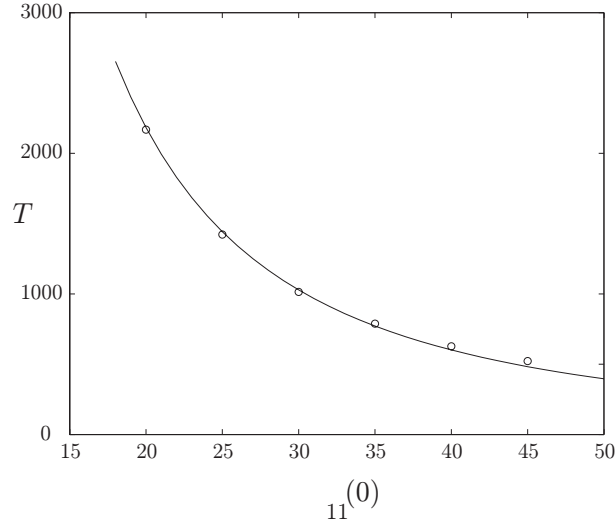


FIGURE 1. Comparison of the results for the period of the two-mode model as a function of $\rho_{11} = \rho_{22}$ with the simulations from the full model (Eq. 3) for parameter values $\rho_{12} = 0.01e^{i0.6}$, $\rho_{21} = 0.01$, $u_{11} = 10^{-3}$, $u_{12} = 0.97 \times 10^{-3}$, $u_{22} = 0.93 \times 10^{-3}$. Solid line: predictions of the two-mode model. Circles: results from simulations of Eqs. (3).

The specific values of $\rho_{jk}(t)$ in particular situations are not in such an excellent quantitative agreement but at least the qualitative behavior is captured by the simple model as seen in Fig. 2. In Fig. 2(a) (upper solid curve) we observe that the two modes used capture most of the norm of the solution, which is consistent with the very good approximation to the period provided by them. However, there are some inaccuracies which also are present in other simulations which we have performed: (i) the instability tends to set in faster in the two-mode model than in the full system and (ii) the fraction of particles transferred is smaller in the PDE model. In any case, even though these details differ, the inter-peak distance which gives the period is very similar between both models, as was expected from the results shown in Fig. 1.

Of course, when the $\rho_{jk}(0)$ or u_{jk} constants are very large, corresponding to a strongly nonlinear situation then the model, which is based on an expansion on the basis of modes of the linear problem, is expected to fail. Another situation where we have found the model to fail is for medium to large perturbation values for which the periods are overestimated by a factor as large as 4.

4.2. Analysis of the two-mode equations. A systematic analysis of the dependence of the period of the solutions of the two mode model is a formidable problem due to the large number of parameters involved. In what follows we study several specific examples.

First, we analyze the dependence of the period on the initial data, specifically on the amount of perturbation (i.e. the values of $\rho_{12}(0), \rho_{21}(0)$) and their phases. From Fig. 3(a) we see that the period depends strongly on the exact amount of perturbation destabilizing the unstable configuration. Also there is a (smaller)

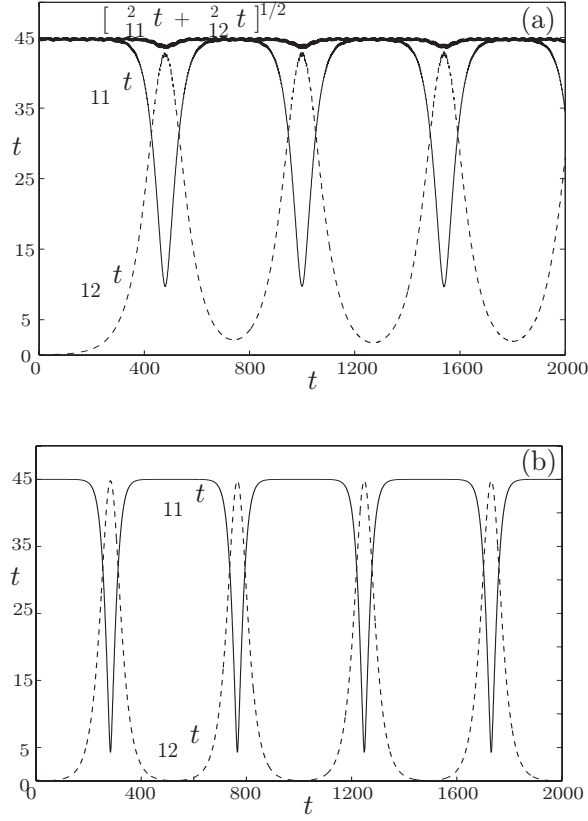


FIGURE 2. Comparison of the results of the two-mode model with the simulations from the full model (Eq. 3) for parameter values $a_{11}(0) = a_{22}(0) = 35, a_{12}(0) = 0.01e^{i0.6}, a_{21} = 0.01, u_{11} = 10^{-3}, u_{12} = 0.97 \times 10^{-3}, u_{22} = 0.93 \times 10^{-3}$. (a) Evolution of $\rho_{11}(t) = |\int \phi_1^*(\mathbf{x})\psi_1(\mathbf{x}, t)d\mathbf{x}|$ (solid line) and $\rho_{12}(t) = |\int \phi_1^*(\mathbf{x})\psi_2(\mathbf{x}, t)d\mathbf{x}|$ (dashed line) for initial data of the type $\psi_1(\mathbf{x}, 0) = a_{11}(0)\phi_1 + a_{12}(0)\phi_2, \psi_2(\mathbf{x}, 0) = a_{21}(0)\phi_1 + a_{22}(0)\phi_2$. Upper solid line is the norm contained in the two basic modes (b) Evolution of $\rho_{11}(t), \rho_{12}(t)$ as predicted by Eq. (56) for the same initial data.

dependence of the period on the phase Φ (Fig. 3(b)). Since in normal experimental situations, the perturbations are not controlled but generated by imperfect initial data, noise, etc., this sensitivity could limit the applicability of the model (3) to predict the specific value of the transfer period. On the other side, by controlling the distance of the initial configuration to the stationary (unstable) one, it is possible to achieve a wide variety of periods for given physical parameters.

We have also studied, as shown in Fig. 4, the dependence of the period as a function of the values of physical constants u_{ij} for different values of the initial data $\rho_{ij}(0)$. The most remarkable feature is the strong increase of the period when u_{22} is much smaller than u_{11} . This phenomenon is due to the small coupling between the two populations, which leads to slower transfers.

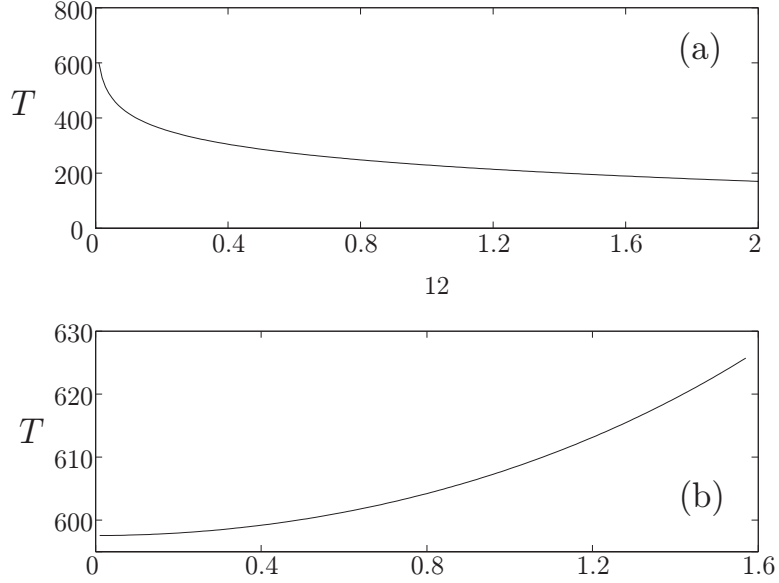


FIGURE 3. Dependence of the period on the variation of initial data. Basic parameters are $u_{11} = 10^{-3}$, $u_{12} = 0.97 \times 10^{-3}$, $u_{22} = 0.93 \times 10^{-3}$, $a_{11}(0) = a_{22}(0) = 40$, $a_{12}(0) = 10^{-2}e^{0.6i}$, $a_{21} = 10^{-2}$. (a) Dependence of the period as a function of the values of a_{12} , a_{21} in the range $\rho_{12}, \rho_{21} \in [0.05, 2]$ (this value corresponds to a perturbation to the norm of the order of 0.2%) (b) Dependence of T as a function of the change of the phase of the perturbation taking $a_{12}(0) = 10^{-2}e^{i\Phi}$, with $\Phi \in [0.01, \pi/2]$

4.3. Other applications. One interesting application of the formulae from the two-mode model is to give the period for small coupling values. Due to the presence of terms of the type $e^{i\mu t}$ for μ of the order of one in the solutions ψ_1, ψ_2 the time step in the numerical simulation cannot be chosen to be too large. On the other hand, for small coupling values the period of the transfer becomes very large (as can be seen for instance in Fig. 1). Thus, the solution has two time scales one small and other very large, which makes it difficult to extract the period from numerical simulations of Eq. (3). In this situation Eqs. (56) and (57) can be a helpful tool to provide results.

Another interesting applications of the two-mode formulae is the analysis of the vortex transfer mechanisms in three-dimensional systems. Again, in such problems the predictive power of Eqs. (3) is limited due to computational limitations. On the other hand the two mode model can be used by just changing the coefficients E_j, C_{jk} by their three-dimensional counterparts.

5. Conclusions. In this paper we have found the exact solution of the two-mode model developed for the explanation of the vortex transfer mechanisms described in Refs. [6, 12, 14]. The most remarkable result is that all the solutions are periodic functions and thus the periodic transfer mechanisms of the vorticity are natural

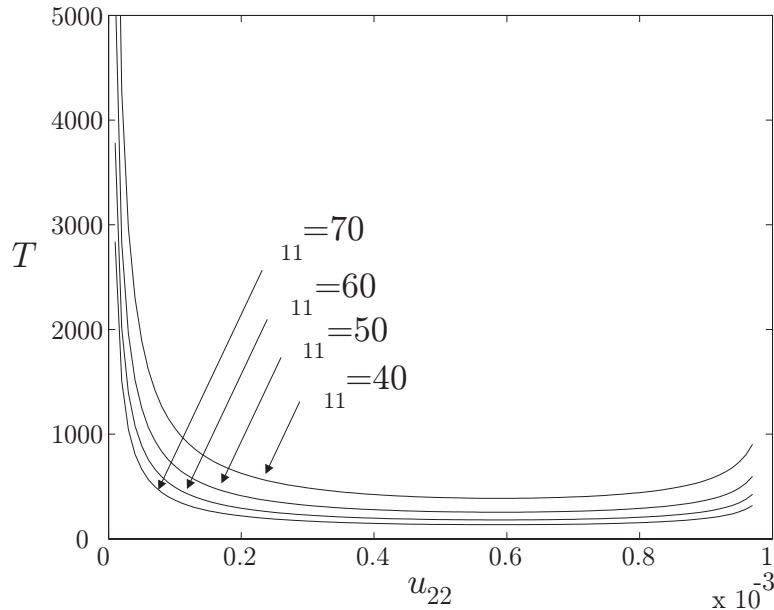


FIGURE 4. Dependence of the period T on u_{ij} for different initial data. Initial data are $a_{11}(0) = [40, 50, 60, 70]$, $a_{12}(0) = 10^{-2}e^{0.6i}$, $a_{21}(0) = 10^{-2}$, $a_{22}(0) = a_{11}(0)$. We choose $u_{11} = 10^{-3}$ and vary u_{22} in the range $[0.01u_{11}, 0.99u_{11}]$. Finally $u_{12} = 2u_{11}u_{22}/(u_{11} + u_{22})$.

within the range of validity of the model, which was previously found to be at least that of pancake traps and some regimes of fully three-dimensional traps.

It has been also shown the relevance of the different parameters on the frequency of the oscillations. In general, this dependence is nontrivial and given by the explicit formulae here found

Acknowledgements. We are grateful to J. J. García-Ripoll for discussions. This work has been partially supported by the Ministerio de Ciencia y Tecnología under grant BFM2000-0521 and Consejería de Ciencia y Tecnología de la Junta de Comunidades de Castilla-La Mancha under grant PAC-02-002. V. Vekslerchik is supported by Ministerio de Educación, Cultura y Deporte under grant SB99-AH777133.

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Received March 2002; revised September 2002; final version January 2003.

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