A Barenblatt's model for Fracture Mechanics

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The functional we consider here, according to Barenblatt' s model of fracture mechanics, describes the energy in the context of fractures appearing under the regime of finite linearized elasticity. It involves explicitly both the tangential component of the fracture - in the spirit of Griffith's theory - and the normal component of the fracture, requiring an infinite amount of energy to produce impenetration, (cf. the third, the second term in (0.1) and the assumptions made of the surface energy density φ (0.2) below).

More precisely, the aim of this work consists of studying the asymptotic behavior of the energy below, by seeking suitable properties on the energy surface density φ,

$$\mathcal{F}(u_h) := \int_\Omega \frac{1}{2}(\mathbb{C}\mathcal{E}u_h, \mathcal{E}u_h)dx + \int_{J_{u_h}} \varphi([u_h]\cdot \nu_{u_h})d\mathcal{H}^{N-1} + C\mathcal{H}^{N-1}(J_{u_h}) \tag{0.1}$$

energyfractu

as $h \to +\infty$, where $\mathbb{C}$ is the Cauchy elasticity tensor, $\mathcal{E}u_h\mathcal{L}^N$ is the absolutely continuous part of the elastic strain and C is a positive constant.

Thus, the first term in the energy (0.1) is a bulk term due to the linear elaticity in the "solid" region, the second term is a surface term, representing the energy released by the normal component of the fracture and a natural requirement consists of asking that the energy becomes infinite if the fracture tries to recompose, i.e. if the sign of $[u_h]\cdot\nu_{u_h} < 0$. On the other hand, if no fracture is present in the sample, the surface energy should be 0. The last term takes into account the energy increase with the measure of the fracture, and involves the tangential component of the fracture.

We will consider an energy density φ of the type

$$\varphi(s) := \begin{cases} +\infty & \text{if } s \in]-\infty, 0[, \\ \geq 0 & \text{if } s \in [0, +\infty[. \end{cases} \tag{0.2}$$

surfaceterm1

By virtue of the by now classical compactness result (see Theorem 1.1. in [5], cf. [3]), a natural requirement is to investigate the lower semicontinuity of the energy (0.1) with respect to the convergences

$$\begin{array}{l} u_h \to u \text{ strongly in } L^1_{\rm loc}(\mathbb{R}^N), \\ \mathcal{E}u_h \rightharpoonup \mathcal{E}u \text{ in } L^2(\Omega; M^{N\times N}_{\rm sym}), \\ E^j u_h \rightharpoonup E^j u \text{ weakly in } \mathcal{M}_b(\Omega; M^{N\times N}_{\rm sym})). \end{array}$$

By assuming that

$$\|u_h\|_{L^\infty(\Omega;\mathbb{R}^N)} + \|\mathcal{E}u_h\|_{L^2(\Omega;M^{N\times N}_{\rm sym})} + \mathcal{H}^{N-1}(J_{u_h}) \leq K \tag{0.3}$$

L2bound

the already mentioned compactness Theorem 1.1. and the lower semicontinuity result (see Corollary 1.2 in [5]) ensure lower semicontinuity with respect to the L^1 strong convergence for the first and third term in the energy (0.1).

In fact our aim consists of finding sufficient conditions on φ to ensure lower semicontinuity of the functional in (0.1), on a suitable class of SBD functions.

Indeed our main result is the following:

mainthm **Theorem 0.1.** *Let Ω be a bounded open subset of $\mathbb{R}^N$. Let $\mathcal{F}(u)$ be the functional defined in (0.1) for every $u \in SBD_2(\Omega)$, where the function $\varphi : \mathbb{R} \to [0, +\infty]$ in (0.2) satisfies the following assumption: there exists a family of indices $\mathcal{A}$ such that*

$$\varphi(s) := \sup_{\substack{\alpha \in \mathcal{A} \\ a_\alpha \in \mathbb{R}_+ b_\alpha \geq 0}} \{(a_\alpha s + b_\alpha)\} \tag{0.4}$$

supfamaffpos

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for every $s \in \mathbb{R}^+$, *where* $\varphi_0(s) = 0$ *for every* $s \in \mathbb{R}^+$. *Let* $\{u_h\}$ *be a sequence in* $SBD_2(\Omega)$, *such that* $[u_h] \cdot \nu_{u_h} \geq 0$ *for every* h *and for every* $x \in J_{u_h}$, *converging to* u *in* $L^1(\Omega;\mathbb{R}^N)$ *satisfying (0.3), Then*

$$\mathcal{F}(u) \leq \liminf_{h\to+\infty} \mathcal{F}(u_h) \tag{0.5}$$ lsc

We remark that our result cannot be obtained by mere extension to SBD of the results already avalaible in the framework of SBV spaces, essentially because SBD spaces require techniques which are different from the SBV analogues. On the other hand the same problem can be addressed also in the SBV framework, leading to comparable results.

Furthermore we provide a carachterization, in terms of Convex Analysis, of the energy densities φ in (0.2) which ensure lower semicontinuity. The following result will be proved:

zazionivarphi **Theorem 0.2.** *Let* $\varphi : \mathbb{R} \to [0 + \infty]$ *be the function defined in (0.2). Furthermore assume that* φ *is continuous in* $\mathrm{dom}(\varphi)$ *and of class* $\mathcal{C}^2$ *in* $\mathrm{int}(\mathrm{dom}(\varphi))$. *Then the following are equivalent:*

1. φ *is defined by (0.4) for some families* $\mathcal{A}$ *for* $s > 0$.
2. φ *is convex, subadditive and nondecreasing.*
3. φ *is convex, nondecreasing and* $\frac{\varphi(\cdot)}{\cdot}$ *is nonincreasing.*
4. *the following two conditions hold:*

$$\varphi^{''}(s) \geq 0 \text{ and } 0 < \varphi'(s) \leq \frac{\varphi(s)}{s} \text{ for every } s \in \mathrm{int}(\mathrm{dom}(\varphi)). \tag{0.6}$$ convsub2

5. *the polar function* φ^* *is not positive in its effective domain, and if* $r \leq 0$, *then* $\varphi^*(r) = \varphi^*(0)$.

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