We establish a new estimate for the topological degree of continuous maps from the sphere \mathbb{S}^N into itself and discuss a optimal constant in this estimate.

The first result is the following (see [3])

Theorem 1 (J. Bourgain, H. Brezis, H-M. Nguyen) Let $g : \mathbb{S}^N \longrightarrow \mathbb{S}^N$ be a continuous function. Then, for every $0 < \delta < \sqrt{2}$, there exists a constant $C = C(\delta, N)$, independent of g, such that

$$\deg g| \le C \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{1}{|x-y|^{2N}} \, dx \, dy. \tag{1}$$

Estimate (1) trivially implies

$$|\text{deg }g| \leq C(p,N) \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{|g(x) - g(y)|^p}{|x - y|^{2N}} \, dx \, dy, \quad \forall \, g \in C(\mathbb{S}^N, \mathbb{S}^N).$$

which was proved in [1]. Estimate (1) was already known for N = 1 (see [2]), but the proof given in [2] was quite involved and could not be extended to higher dimensions. Moreover, we will show that (1) holds if and only if

$$0 < \delta < \ell_N := \sqrt{2 + \frac{2}{N+1}}$$

More precisely (see [4]),

Theorem 2 (*H-M. Nguyen*) Let $g \in C(\mathbb{S}^N, \mathbb{S}^N)$, $\delta \in (0, \ell_N)$. Then

$$\left|\deg g\right| \le C \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{1}{|x-y|^{2N}} \, dx \, dy,\tag{2}$$

for some positive constant $C = C(\delta, N)$.

Furthermore, there exists a sequence $\{g_k\}_{k\in\mathbb{N}}\subset C(\mathbb{S}^N,\mathbb{S}^N)$ such that

$$\deg g_k = 1, \quad \forall k \ge 1, \tag{3}$$

and

$$\lim_{k \to \infty} \int_{\mathbb{S}^N} \int_{\mathbb{S}^N} \frac{1}{|x - y|^{2N}} \, dx \, dy = 0.$$
(4)

Références

- J.Bourgain, H.Brezis, P.Mironescu, Lifting, Degree, and Distributional Jacobian Revisited, Comm. Pure Appl. Math., 58 (2005), 529-551.
- [2] J.Bourgain, H.Brezis, P.Mironescu, Complements to the paper :"Lifting, Degree, and Distributional Jacobian Revisited", to be posted on the website http://ann.jussieu.fr/publications.
- [3] J.Bourgain, H.Brezis, H-M.Nguyen, A new estimate for the topological degree, C. R. Acad Sci. Paris, Ser. I 340(2005), 787-791.
- [4] H-M.Nguyen In preparation.