Local Markets and Rational Expectations*

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Abstract

Our objective is to develop a framework where consumers make optimal consumption and location decisions in a spatial economy. Local markets are assumed perfectly competitive and migration is modelled as an investment decision. We show that the location problem is a problem in the calculus of variations. Existence of a solution is proved under mild conditions.

Keywords: local markets, migration, expectations, spatial economy

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Our objective is to develop a framework where consumers make optimal consumption and location decisions in a spatial economy. We consider a spatial economy with a continuum of locations distributed along a circle, $s \in \mathbb{S} = [0, 2\Pi]$. Time is denoted by $t \in [t_0, T]$. There are H types of agents denoted by h, who are distributed along the real line $\mathbb{S}$. The density of type-h agents in location s, at time t, is denoted by $A^h(s,t)$. There are L local goods denoted by l. Each type-h agent is continuously endowed with a constant bundle $\mathbf{e}^h \in \mathbb{R}^L_+$ over time, and cares about sequences of consumption $\mathbf{X}^h(t) \in \mathbb{R}^L$. The instantaneous utility function represented by $U(.)$ is the same for all agents.

First we study how consumption and migration decisions are simultaneously taken in terms of given intertemporal prices - that is in terms of a spatial distribution of prices which evolves over time. Consider a type-h agent initially located in s_0^h at time t_0. His problem is

$$\max_{s^h(t), \mathbf{X}^h(t)} \int_{t_0}^{T} \left\{ U\left[\mathbf{X}^h(t)\right] - \frac{1}{2k^h}\left[\frac{ds^h}{dt}(t)\right]^2 \right\} dt$$

$$\text{st.} \quad \mathbf{p}(s^h(t), t) \bullet \left[\mathbf{X}^h(t) - \mathbf{e}^h\right] = 0, \forall t \in [t_0, T]$$

$$\text{with } s^h(t_0) = s_0^h \tag{$\mathbb{P}$}$$

where k^h measures the migration cost incurred by a type-h agent.

The consumption problem is

$$\max_{\mathbf{X}^h(t)} U\left[\mathbf{X}^h(t)\right]$$

$$\text{st.} \quad \mathbf{p}(s^h(t), t) \bullet \left[\mathbf{X}^h(t) - \mathbf{e}^h\right] = 0 \tag{$\mathbb{P}_1$}$$

The location problem can be written as

$$\max_{s(t)} J[s] = \max_{s(t)} \int_{t_0}^{T} \left\{ U(s(t), t) - \frac{1}{2k}\dot{s}^2(t) \right\} dt \tag{$\mathbb{P}_2$}$$

$$\text{with } s(t_0) = s_0$$

where $U(s(t), t)$ denotes $U\left[\mathbf{X}^h[\mathbf{p}(s^h(t), t), \mathbf{e}^h]\right]$.

Proposition 1 *If the price $\mathbf{p}(s(t), t)$ is continuous and the utility function U is bounded from above, then a solution $\overline{s}$ to ($\mathbb{P}_2$) exists in the set $H^1\left(]t_0, T[\right)$.*

The proof is in five steps, see Dacorogna (1992) or Buttazo *and al.* (1998).

(i) First we fix the class of admissible functions to the Sobolev space $W^{1,2}\left(]t_0,T[\right)$ and the notion of weak convergence which is denoted by $s_n \rightharpoonup s$.

(ii) Next we show that the functional J is well defined on $W^{1,2}$ and bounded from above, so that $\sup J$ is finite. This implies that we can find a maximizing sequence in $W^{1,2}$, ie. a sequence of functions $s_n \in W^{1,2}$, $n = 1, 2, ...$ such that $J(s_n) \longrightarrow \sup_{W^{1,2}}$ J.

(iii) Then we prove that J is sequentially upper semi continuous on $W^{1,2}$ with respect to the weak convergence. That is, we have to verify that $s_n \rightharpoonup \overline{s}$ implies $J(\overline{s}) \geq \limsup\limits_{n\to\infty} J\left(s_n\right)$.

(iv) Finally we show the sequential compactness of $W^{1,2}$ with respect to the weak convergence. In fact, it suffices to prove that any - or at least one - maximizing sequence contains a convergent subsequence with a limit $\overline{s}$ in $W^{1,2}$.

(v) Then as $\{s_n\}$ is a maximizing sequence, $\overline{s}$ is necessarily a solution to $(\mathbb{P}_2)$.

Theorem 1 (Weak Euler-Lagrange) *Suppose $U(s(t),t)$ is C^1, $|\partial_s U| \leq \delta(s)$, $\delta(s)$ C^0, $\delta'(s) \geq 0$. If $\overline{s}(t) \in W^{1,2}$ is a maximizer of $J[s]$, with $\overline{s}(t_0) = s_0$, then $\overline{s}(t)$ satisfies the weak form of the Euler-Lagrange equation,*

$$\int_{t_0}^{T} \left(\partial_s U(\overline{s}(t),t)\varphi - \frac{1}{k}\dot{\overline{s}}\dot{\varphi} \right) dt = 0 \ , \ \forall \varphi \in W^{1,2} : \varphi(t_0) = 0$$

Theorem 2 (Euler-Lagrange) *Suppose $U(s,t)$ is C^1 with respect to s. If $\overline{s}(t) \in W^{1,2}$ is a maximizer of $J[s]$, with $\overline{s}(t_0) = s_0$, then $\overline{s}(t)$ satisfies the strong form of the Euler-Lagrange equation,*

$$\ddot{\overline{s}} = -k\partial_s U(\overline{s}(t),t)$$

Theorem 3 (Regularity) *Suppose $U(s,t)$ is C^k with respect to s, $k \geq 1$. If $\overline{s}(t) \in W^{1,2}$ is a maximizer of $J[s]$, with $\overline{s}(t_0) = s_0$, then $\overline{s}(t) \in C^{k+1}$.*

References

[1] Buttazzo, G., Giaquinta, M. and S. Hildenbrandt (1998), "One-dimensional Variational Problems - An Introduction," Oxford Science Publications.

[2] Dacorogna, B. (1992), "Intoduction au Calcul des Variations," Cahiers Mathématiques de l'Ecole Polytechnique Fédérale de Lausanne, *Presses polytechniques et universitaires romandes.*

[3] Ekeland, I. (1986), "Some Variational Problems from Mathematical Economics", in A. Ambrosetti and R. Lucchetti (eds), *Mathematical Economics*, Lectures Notes in Mathematics, Springer-Verlag.

[4] Mossay, P. (2006), Spatial Stability, *Regional Science and Urban Economics*, forthcoming.

[5] Sonnenschein, H. (1981), "Price Dynamics and the Disappearance of Short Run Profits: An Example", *Journal of Mathematical Economics*, 8, 201-204.

[6] Sonnenschein, H. (1982), "Price Dynamics Based on the Adjustment of Firms," *American Economic Review*, 72, 1088-1096.