Approximation of Hölder homeomorphisms by piecewise affine homeomorphisms

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This talk is concerned with the problem of approximating a homeomorphism by piecewise affine homeomorphisms.

Consider an elastic body whose particles are labelled by the position that they occupy in a reference configuration $\Omega \subset \mathbb{R}^n$. Typically, $n \in \{2, 3\}$. A deformation of the body is described by a mapping $y : \Omega \to \mathbb{R}^n$, where $y(x)$ denotes the deformed position of the material point $x \in \Omega$. It is natural to assume that y belongs to the Sobolev space $W^{1,1}(\Omega, \mathbb{R}^n)$. Every equilibrium solution must be a critical point of the elastic energy

$$I(y) := \int_\Omega W(Dy(x))dx,$$

where $W : \mathbb{R}^{n\times n} \to [0, \infty]$ is the stored-energy function of the material. Usually, one looks for minima of the functional I. The stored-energy function W should be continuous, and satisfy certain coercivity conditions, generalized convexity conditions, rotational invariance, and $W(A) = \infty$ whenever $\det A \leq 0$ (see Ball [2]). In addition, this deformation must satisfy the following two requirements, in order to be physically realistic:

C1 y must preserve the orientation; mathematically, $\det Dy > 0$ a.e.

C2 interpenetration of matter must not occur; mathematically, y must be invertible.

When one approximates by finite elements an equilibrium solution in nonlinear elastostatics, it is important that those approximations also satisfy conditions **C1** and **C2**.

The possibility of approximating a homeomorphism in the supremum norm by piecewise linear homeomorphisms was proved by Moise [4] in the case $n \in \{2, 3\}$. However, an important open question (see Ball [1]) is the approximation of a $W^{1,p}$ function by piecewise affine homeomorphisms in the $W^{1,p}$ norm.

In this work we consider this problem in dimension 2 and approximate in the Hölder norm. Specifically, our main result is the following.

Theorem *Let $\Omega \subset \mathbb{R}^2$ be an open set with polygonal boundary. Let $0 < \alpha \leq 1$. Let $h \in C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^2)$ be a homeomorphism such that $h^{-1} \in C^{0,\alpha}(h(\bar{\Omega}), \mathbb{R}^2)$. Then there exists $0 < \beta < \alpha$ such that for each $\varepsilon > 0$ there exists a piecewise affine homeomorphism $f : \bar{\Omega} \to \mathbb{R}^2$ with*

$$\|f - h\|_{C^{0,\beta}} < \varepsilon.$$

Here $C^{0,\alpha}$ is the Banach space of Hölder continuous functions of exponent α.

Our proof adapts Moise's [4] and constructs an explicit triangulation of Ω, which, when $\alpha = 1$, is regular in the sense of Ciarlet [3].

References

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