A characterization of convex calibrable sets in $\mathbf{R}^N$ with respect to anisotropic norms

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Let F be a convex subset of $\mathbf{R}^2$ and let ϕ be an anisotropy. It has been proved in [3] that the following three assertions are equivalent.

(a) F is ϕ-calibrable, i.e., $\exists\, \xi \in L^\infty(F, \mathbf{R}^2)$, with $\phi(\xi(x)) \leq 1$ a.e. in F (where ϕ is the dual norm of ϕ^0), such that

$$\begin{aligned} -\operatorname{div} \xi &= \lambda_F^\phi := \frac{P_\phi(F)}{|F|} \quad \text{in } F, \\ \xi \cdot \nu^F &= -\phi^0(\nu^F) \quad \text{in } \partial F, \end{aligned} \tag{1}$$

where

$$P_\phi(F) := \int_{\partial F} \phi^0(\nu^E)\, d\mathcal{H}^{N-1}.$$

Here, ν^F is the outward unit normal to the boundary ∂F of F and ϕ^0 (the surface tension) is a norm on $\mathbf{R}^2$.

(b) F is a solution of the problem

$$\min_{X \subseteq F} \; P_\phi(X) - \lambda_F^\phi |X|. \tag{2}$$

(c) We have

$$\operatorname*{ess\,sup}_{x \in \partial F} \; \kappa_F^\phi(x) \leq \lambda_F^\phi\,, \tag{3}$$

where $\kappa_F^\phi(x)$ denotes the anisotropic curvature of ∂F at x.

The characterization of the calibrability of a convex set in $\mathbf{R}^2$, with respect to the euclidean perimeter was proved by Giusti in [4] and was rederived in [2] where calibrable sets were used to construct explicit solutions of the denoising problem in image processing. The extension of the above result for the euclidean perimeter and $N \geq 3$ was proved in [1]. In this work we extend the above set of equivalences to the anisotropic case, for a convex set in $\mathbf{R}^N$ which satisfies a ball condition.

As an interesting by-product of our analysis we obtain that solutions of

$$\min_{X \subseteq C, |X| = V} P_\phi(X) \tag{4}$$

where $0 < V < |C|$ are convex sets for $V \in [|K|, |C|]$ where K is a convex ϕ-calibrable set contained in C.

Our results also enable us to describe the evolution of any convex set in $\mathbf{R}^N$, satisfying a ball condition, by the minimizing anisotropic total variation flow.

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