Quasiconvexity: the quadratic case revisited, and some consequences for fourth-degree polynomials

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Abstract

We provide an alternative proof for the well-known quadratic case for which quasiconvexity and rank-one convexity are equivalent, which does not make use of Plancherel formula. Some consequences of the same ideas for the case of 4th degree homogeneous polynomials are shown.

Keywords: Quasiconvexity, rank-one convexity, sequential weak lower semicontinuity.

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1 Introduction

The central problem in the Calculus of Variations is that of showing existence of minimizers of energy functionals of the type

\[ I(u) = \int_\Omega \varphi(\nabla u(x)) \, dx \]

among competing fields \( u : \Omega \subset \mathbb{R}^N \to \mathbb{R}^m \) complying with boundary conditions over \( \partial \Omega \) ([8]). Here \( \Omega \) is supposed to be a bounded, regular domain (i.e. Lipschitz), and feasible fields \( u \) belong to suitable Sobolev classes related to the growth properties of the density \( \varphi \) at infinity. The integrand \( \varphi : \mathbb{R}^{m \times N} \to \mathbb{R} \) is assumed to be continuous. More specific assumptions are necessary to deal with problems in non-linear elasticity ([7]).

The crucial property on \( \varphi \) to ensure existence of solutions through the direct method ([8]) is the quasiconvexity. One such density \( \varphi \) is said to be quasiconvex if

\[ \varphi(\xi) \leq \int_D \varphi(\xi + \nabla v(x)) \, dx \]  

(1)
for some unitary domain \( D \) (\(|D| = 1\)), for any matrix \( \xi \in \mathbb{M}^{m \times N} \), and every test field \( v \) in \( D \). It turns out that this concept is independent of the domain. This property on \( \varphi \) is equivalent to the weak lower semicontinuity of the functional \( I \) above with respect to weak convergence of Lipschitz fields. This was established by Morrey in [15].

In the scalar case, when \( m = 1 \) or \( N = 1 \), quasiconvexity reduces to plain convexity, but it is not so in the fully vector case \( N, m > 1 \). The concept of quasiconvexity is hard to grasp and analyze due to its non-local character expressed in the inequality (1) above. So a principal issue has been to find more manageable necessary and sufficient conditions for it.

A main necessary condition is rank-one convexity. An integrand \( \varphi \) as before is rank-one convex if it satisfies the typical convexity inequality along rank-one matrices

\[
\varphi(t\xi_1 + (1-t)\xi_0) \leq t\varphi(\xi_1) + (1-t)\varphi(\xi_0), \quad \text{rank}(\xi_1 - \xi_0) \leq 1, \quad t \in [0,1].
\]

While a sufficient condition is polyconvexity. \( \varphi \) is polyconvex if it can be written in the form

\[
\varphi(\xi) = \phi(M(\xi))
\]

where \( M(\xi) \) is the vector of all minors of \( \xi \), and \( \phi \) is a convex function of all its arguments. Polyconvexity has played a major role in existence theorems in non-linear elasticity ([2]). A lot of effort has been dedicated to establishing the differences among these three convexity concepts. All three are different and counterexamples of various forms have been found through the years (see [1], [9], [21], [23]).

The equivalence between rank-one convexity and quasiconvexity is the one that has stood unsolved longer. Morrey himself ([16]) stated that “it is an unsolved problem to prove or disprove the theorem that every rank-one convex function of \( \nabla u \) is quasiconvex.” In his seminal paper [15], he conjectured (informally) that “… after a great deal of experimentation, the writer is inclined to think that there is no condition of the type discussed, which involves \( \varphi \) and only a finite number of its derivatives, and which is both necessary and sufficient for quasiconvexity in the general case.” So, usually, Morrey’s conjecture is stated by saying that rank-one convexity does not imply quasiconvexity.

In the special class of quadratic forms, it was known long ago, and not difficult to see through Fourier analysis (making use of the Plancherel formula), that these two kinds of convexity are equivalent, and evidence in favor of the equivalence and against it started to pile up (see [3] for a very nice account of the situation until 1986) until the conclusive counterexample of Šverák ([22]). As far as we can tell, there is no essentially new counterexample, and that one is only valid for \( m \geq 3 \) so that rank-one convexity does not imply quasiconvexity in this situation. Further attempts to extend the counterexample for \( m = 2 \) have failed ([20]).

Some additional evidence against the equivalence can be found in [19], while evidence in favor is contained in [6] and [17]. The problem remains open for \( m = 2 \).

In this work we provide an alternative proof for the quadratic case (Section 2), which no longer makes use of Plancherel formula, an so it can, in principle, be applied to other cases, especially to polynomials. This
has further interest nowadays, as we now know that one can approximate quasiconvex functions by quasiconvex polynomials ([13]).

In Section 3 we apply our techniques developed for the quadratic case to derive conditions that are both necessary and sufficient for quasiconvexity at the origin in the case of the fourth degree homogeneous polynomials. This conditions, however, are not enough transparent and so in Section 4 we apply this tools to the case of quasiconvexity for second gradients, usually called 2-quasiconvexity. A function \( \varphi : I_M^{N \times N} \to \mathbb{R} \) is said to be 2-quasiconvex if

\[
\varphi(\xi) \leq \int_D \varphi(\xi + \nabla^2 v(x)) \, dx,
\]

for any \( \xi \in I_M^{N \times N} \) and every \( v \in C^\infty_c(D, \mathbb{R}) \), where \( I_M^{N \times N} \) denote the space of \( I_M^{N \times N} \)-symmetric matrices and \( |D| = 1 \) (the choice of the domain is irrelevant, if it is a bounded set of \( \mathbb{R}^N \), [14]).

Quasiconvexity for second order gradients is a particular case of high-order quasiconvexity, introduced by Meyers ([14]), and it is the natural notion of quasiconvexity for high-order variational problems. An example of such problems are the gradient theories of phase transitions within elasticity regimes (see e.g. [5]). Despite the fact that in [11] (generalizing a result from [18]) it was proved that 2-quasiconvexity reduces to quasiconvexity for symmetric matrices or, to be more precise, that each 2-quasiconvex function is the restriction of a quasiconvex function to the space of symmetric matrices, the interest here is the simplification of the computations involved, yielding more usable first order conditions, which gives the possibility of treating explicitly some examples.

2 The quadratic case

A well-known result is the following

**Theorem 1** Let \( \varphi : I_M^{m \times N} \to \mathbb{R} \) be a quadratic form. Then \( \varphi \) is quasiconvex if and only if is rank-one convex.

The proof of this result is known for a long time ([24],[25], although implicitly known earlier). Nevertheless, all known proofs until now use Fourier transforms and the Plancherel formula, and so they cannot be applied to other than the quadratic case. We propose an alternative proof, which does not make use of Plancherel formula, and so we hope in this way to gain more insight about this outstanding problem of if rank-one convexity implies quasiconvexity when \( m = 2 \).

**Proof.**

Step 1. Notice that a quadratic form \( \varphi : I_M^{m \times N} \to \mathbb{R} \) can always be written as

\[
\varphi(\xi) = \xi^T A \xi - \xi^T B \xi,
\]

with \( A, B \in I_M^{(m \times N) \times (m \times N)} \) symmetric matrices and \( A \) is positive definite.

By definition, \( \varphi \) is quasiconvex if it verifies

\[
\int_Q \varphi(\xi + \nabla u(x)) \, dx \geq \varphi(\xi),
\]
for every $\xi \in \mathbb{M}^{m,N}$, $u \in C^{\infty}_0(Q, \mathbb{R}^m)$, where $Q = (0,1)^N$. For $\varphi$ as above, we write the above inequality as

$$
\int_Q (\xi + \nabla u(x))^T A (\xi + \nabla u(x)) - (\xi + \nabla u(x))^T B (\xi + \nabla u(x)) \, dx \geq \xi^T A \xi - \xi^T B \xi \\
\iff \int_Q \nabla^T u(x) A \nabla u(x) \, dx \geq \int_Q \nabla^T u(x) B \nabla u(x) \, dx \iff
$$

$$
1 \geq \frac{\int_Q \nabla^T u(x) B \nabla u(x) \, dx}{\int_Q \nabla^T u(x) A \nabla u(x) \, dx},
$$

for every $u \in C^{\infty}_0(Q, \mathbb{R}^m)$ since, by the divergence theorem,

$$
\int_Q \nabla u(x) \, dx = 0.
$$

This last inequality can be rewritten as

$$
1 \geq \max_u \frac{\int_Q \nabla^T u(x) B \nabla u(x) \, dx}{\int_Q \nabla^T u(x) A \nabla u(x) \, dx},
$$

for $u \in C^{\infty}_0(Q, \mathbb{R}^m)$.

**Step 2.** We want now to solve the (infinite dimensional) problem of finding

$$
\max_u \frac{\int_Q \nabla^T u(x) B \nabla u(x) \, dx}{\int_Q \nabla^T u(x) A \nabla u(x) \, dx},
$$

where $A$ is positive definite. However, we can reduce this infinite dimensional problem to a finite dimensional one (but now with an infinite number of variables), by expanding $u$ in a Fourier series. If $u \in C^{\infty}_0(Q, \mathbb{R}^m)$, we put

$$
u(x) = \sum_{k \in \mathbb{Z}^N} c_k e^{2\pi i k \cdot x}, \quad c_k = \int_Q u(x) e^{-2\pi i k \cdot x} \, dx.
$$

Notice that, although we take $k \in \mathbb{Z}^N$ in the summations, we are thinking only in expansions with a finite (but arbitrary) number of terms. In this way it is straightforward to justify all the computations done, and then to achieve the conclusion for any $u \in C^{\infty}_0(Q, \mathbb{R}^m)$ is just a limit procedure to obtain expansions with all $k \in \mathbb{Z}^N ([26])$, preserving in this way required inequality of quasiconvexity. The same assumption is made through the following sections in all the computations involving Fourier expansions.
Now, \( \int_Q \nabla^T u(x) A \nabla u(x) \, dx \) is equal to
\[
-4\pi^2 \int_Q \sum_{k \in \mathbb{Z}^N} (c_k \otimes k)^T e^{2\pi i k \cdot x} A \sum_{j \in \mathbb{Z}^N} (c_j \otimes j) e^{2\pi i j \cdot x} \, dx =
\]
\[
= -4\pi^2 \sum_{k \in \mathbb{Z}^N} \sum_{j \in \mathbb{Z}^N} k \otimes c_k A c_j \otimes j \int_Q e^{2\pi i (k+j) \cdot x} \, dx =
\]
\[
= -4\pi^2 \sum_{k \in \mathbb{Z}^N} k \otimes c_k A c_{-k} \otimes -k =
\]
\[
= 4\pi^2 \sum_{k \in \mathbb{Z}^N} k \otimes c_k A \bar{c}_k \otimes k,
\]
where \( \bar{c}_k \) denotes the complex conjugate of \( c_k \). We can ignore any multiplicative constants, as they will appear both in the numerator and denominator, so we can take
\[
\int_Q \nabla^T u(x) A \nabla u(x) \, dx = \sum_{k \in \mathbb{Z}^N} k \otimes c_k A \bar{c}_k \otimes k.
\]
If we put
\[
c_k = (c^1_k, c^2_k, ..., c^m_k), \; k = (k_1, k_2, ..., k_N),
\]
then
\[
c_k \otimes k = (c^1_k, c^2_k, ..., c^m_k) \otimes (k_1, k_2, ..., k_N) =
\begin{pmatrix}
c^1_k k_1 \\
\vdots \\
c^m_k k_N
\end{pmatrix},
\]
\[
k \otimes \bar{c}_k = \begin{pmatrix}
\bar{c}^1_k k_1 & \cdots & \bar{c}^1_k k_N \\
\bar{c}^2_k k_1 & \cdots & \bar{c}^2_k k_N \\
\vdots & \ddots & \vdots \\
\bar{c}^m_k k_1 & \cdots & \bar{c}^m_k k_N
\end{pmatrix}
\]
and, if we consider

\[
A = \begin{pmatrix}
A_1^1 & A_2^1 & \cdots & A_N^1 & A_{N+1}^1 & \cdots & A_{2N}^1 & \cdots & A_{mN}^1 \\
A_1^2 & A_2^2 & \cdots & A_N^2 & A_{N+1}^2 & \cdots & A_{2N}^2 & \cdots & A_{mN}^2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
A_1^{N+1} & A_2^{N+1} & \cdots & A_N^{N+1} & A_{N+1}^{N+1} & \cdots & A_{2N}^{N+1} & \cdots & A_{mN}^{N+1} \\
A_1^{2N} & A_2^{2N} & \cdots & A_N^{2N} & A_{N+1}^{2N} & \cdots & A_{2N}^{2N} & \cdots & A_{mN}^{2N} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
A_1^{mN} & A_2^{mN} & \cdots & A_N^{mN} & A_{N+1}^{mN} & \cdots & A_{2N}^{mN} & \cdots & A_{mN}^{mN}
\end{pmatrix}
\]

then

\[
\int_Q \nabla^T u(x) A \nabla u(x) \, dx = \sum_{k \in \mathbb{Z}^N} k \otimes c_k \, A \, \bar{c}_k \otimes k = \sum_{k \in \mathbb{Z}^N} c_k^T A_k \, \bar{c}_k,
\]

for

\[
A_k = \begin{pmatrix}
\alpha_1^1(k) & \alpha_2^1(k) & \cdots & \alpha_m^1(k) \\
\alpha_1^2(k) & \alpha_2^2(k) & \cdots & \alpha_m^2(k) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^m(k) & \alpha_2^m(k) & \cdots & \alpha_m^m(k)
\end{pmatrix},
\]

where

\[
\alpha_p^p(k) = \sum_{r=1}^{N} (k_r)^2 A_{r+(p-1)N}^{r+(p-1)N} + 2 \sum_{r,s=1}^{N} k_r k_s A_{r+(p-1)N}^{r+(p-1)N} \, A_{s+(p-1)N}^{r+(p-1)N}, \quad p = 1, \ldots, m,
\]

\[
\alpha_q^p(k) = \sum_{r=1}^{N} (k_r)^2 A_{r+(q-1)N}^{r+(q-1)N} + \sum_{r,s=1}^{N} k_r k_s \left( A_{r+(q-1)N}^{r+(q-1)N} + A_{s+(p-1)N}^{r+(p-1)N} \right), \quad p, q = 1, \ldots, m, \quad q < p.
\]

But

\[
\sum_{k \in \mathbb{Z}^N} c_k^T A_k \, \bar{c}_k = \sum_{k \in \mathbb{Z}^N} (\text{Re} \, c_k + i \text{Im} \, c_k)^T A_k \, (\text{Re} \, c_k - i \text{Im} \, c_k) = \sum_{k \in \mathbb{Z}^N} X_k^T \bar{A}_k \, X_k,
\]

for

\[
X_k = \begin{pmatrix}
\text{Re} \, c_k \\
\text{Im} \, c_k
\end{pmatrix}, \quad \bar{A}_k = \begin{pmatrix}
A_k & 0 \\
0 & A_k
\end{pmatrix}.
\]
Similarly, we have

\[
\int_Q \nabla^T u(x) B \nabla u(x) \, dx = \sum_{k \in \mathbb{Z}^N} k \otimes c_k B \tau_k \otimes k = \sum_{k \in \mathbb{Z}^N} X_k^T \tilde{B}_k X_k.
\]

Step 3. Determining

\[
\max_{\nabla u} \frac{\int_Q \nabla^T u(x) B \nabla u(x) \, dx}{\int_Q \nabla^T u(x) A \nabla u(x) \, dx},
\]

is equivalent of finding the

\[
\max_{X = (X_k)_{k \in \mathbb{Z}^N}} \frac{\sum_{k \in \mathbb{Z}^N} X_k^T \tilde{B}_k X_k}{\sum_{k \in \mathbb{Z}^N} X_k^T \tilde{A}_k X_k},
\]

which is the maximum of the quotient of two expressions homogeneous of degree two in \( X \). Instead of computing

\[
\max_{X = (X_k)_{k \in \mathbb{Z}^N}} \frac{\sum_{k \in \mathbb{Z}^N} X_k^T \tilde{B}_k X_k}{\sum_{k \in \mathbb{Z}^N} X_k^T \tilde{A}_k X_k},
\]

we can consider the equivalent problem

\[
\max_{X = (X_k)_{k \in \mathbb{Z}^N}} \sum_{k \in \mathbb{Z}^N} X_k^T \tilde{B}_k X_k
\]

subject to

\[
\sum_{k \in \mathbb{Z}^N} X_k^T \tilde{A}_k X_k = 1,
\]

where \( \tilde{A}_k > 0 \) for every \( k \neq 0 \) (all eigenvalues of \( A \) are strictly positive).

If \( \lambda \) is a Lagrange multiplier, we put

\[
L(X, \lambda) = \sum_{k \in \mathbb{Z}^N} X_k^T \tilde{B}_k X_k - \lambda \left( \sum_{k \in \mathbb{Z}^N} X_k^T \tilde{A}_k X_k - 1 \right).
\]

The first-order necessary conditions will then tell us that

\[
\nabla L = 0 \iff \begin{cases} (\tilde{B}_k - \lambda \tilde{A}_k) X_k = 0 \text{ for every } k \in \mathbb{Z}^N, \\ \sum_{k \in \mathbb{Z}^N} X_k^T \tilde{A}_k X_k = 1. \end{cases}
\]

If \( X = (X_k)_{k \in \mathbb{Z}^N} \) is a critical point, then it’s easy to derive

\[
\sum_{k \in \mathbb{Z}^N} X_k^T \tilde{B}_k X_k = \lambda.
\]

Also from the above, for having solutions of this system of equations, it is necessary (and also sufficient, as \( \tilde{A}_k > 0 \)) to have

\[
\det(\tilde{B}_j - \lambda \tilde{A}_j) = 0,
\]
for some \( j \) and some \( \lambda \). The solutions \( \lambda \) of this equation will be denoted by \( \lambda_j \). We put \( X_j = w_j \neq 0 \) and suppose, without loss of generality, that \( X_k = 0 \) for \( k \neq j \). Then

\[
k \neq j \Rightarrow (\tilde{B}_k - \lambda_j \tilde{A}_k) X_k = (\tilde{B}_k - \lambda_j \tilde{A}_k) 0 = 0.
\]

As \( w_j \in \ker(\tilde{B}_j - \lambda_j \tilde{A}_j) \),

\[
(\tilde{B}_j - \lambda_j \tilde{A}_j) X_j = (\tilde{B}_j - \lambda_j \tilde{A}_j) w_j = 0.
\]

With respect to this last equation, we have

\[
\sum_{k \in \mathbb{Z}^N} X_k^T A_k X_k = w_j^T A_j w_j = c^2 > 0.
\]

By setting

\[
s_j = \frac{1}{c} w_j = \frac{1}{c} X_j,
\]

and

\[
s_k = 0, \ k \neq j,
\]

we have that \( ((s_k)_{k \in \mathbb{Z}^N}, \lambda_j) \) is a critical point of the problem, with associated cost equal to \( \lambda_j \).

The above proves that the problem of finding

\[
\max_{X = (X_k)_{k \in \mathbb{Z}^N}} \sum_{k \in \mathbb{Z}^N} X_k^T \tilde{B}_k X_k
\]

subject to

\[
\sum_{k \in \mathbb{Z}^N} X_k^T \tilde{A}_k X_k = 1,
\]

is equivalent to determining

\[
\sup_{j \in \mathbb{Z}^N} \lambda_j,
\]

where \( \lambda_j \) are the solutions of

\[
det(\tilde{B}_j - \lambda_j \tilde{A}_j) = 0, \ j \in \mathbb{Z}^N.
\]

Step 4. Suppose, by hypothesis, that \( \varphi \) is rank one convex. Since \( \varphi \) is smooth, this is equivalent to the Legendre-Hadamard condition

\[
\n \otimes a \nabla^2 \varphi(\xi) a \otimes n \geq 0,
\]

for every \( \xi \in \mathbb{R}^{m \times N}, a \in \mathbb{R}^m, n \in \mathbb{R}^N \). Making the decomposition \( \varphi = \varphi_1 - \varphi_2, \) with \( \varphi_1 \) strictly convex and putting \( \nabla^2 \varphi_1(\xi) = A, \nabla^2 \varphi_2(\xi) = B \) (since \( \varphi \) is quadratic), the above inequality is then equivalent to

\[
1 \geq \frac{n \otimes a B a \otimes n}{n \otimes a A a \otimes n}.
\]
for every $a \in \mathbb{R}^m$ and $n \in \mathbb{R}^N$, which is the same,

$$1 \geq \max_{a \in \mathbb{R}^m, n \in \mathbb{R}^N} \frac{n \otimes a \, B \, a \otimes n}{n \otimes a \, A \, a \otimes n}.$$ 

We can incorporate, in the above maximum, the dependence on $n$ within the matrices $A$ (also in the spirit of the above case for quasiconvexity), thus obtaining

$$n \otimes a \, A \, a \otimes n = a^T A_n \, a$$

with

$$A_n = \begin{pmatrix} \alpha_1^1(n) & \ldots & \alpha_m^1(n) \\ \vdots & \ddots & \vdots \\ \alpha_1^m(n) & \ldots & \alpha_m^m(n) \end{pmatrix},$$

for $\alpha^j_i$, $\alpha^i_j$ defined as before (but now as functions of $n \in \mathbb{R}^N$). Similarly we can use the same reasoning with $B$, thus obtaining

$$n \otimes a \, B \, a \otimes n = a^T B_n \, a.$$ 

We now want to compute

$$\max_{n \in \mathbb{R}^N} \max_{a \in \mathbb{R}^m} \frac{a^T B_n \, a}{a^T A_n \, a}.$$ 

Since

$$\max_{a \in \mathbb{R}^m} \frac{a^T B_n \, a}{a^T A_n \, a}$$

is the quotient of two expressions homogeneous of degree two in $a$, we consider the equivalent problem

$$\max_{a \in \mathbb{R}^m} a^T B_n \, a$$

subject to

$$a^T A_n \, a = 1,$$

where $A_n > 0$ for every $n \neq 0$.

If $\lambda$ is a Lagrange multiplier, we put

$$L(a, \lambda) = a^T B_n \, a - \lambda \left( a^T A_n \, a - 1 \right).$$

The first-order necessary conditions will then tell us that

$$\nabla L = 0 \iff \begin{cases} (B_n - \lambda A_n) \, a = 0 \\ a^T A_n \, a = 1. \end{cases}$$

Once again, if $a \in \mathbb{R}^m$ is a critical point, then

$$a^T B_n \, a = \lambda.$$ 

The above system has solutions if and only if $\det(B_n - \lambda A_n) = 0$. The solutions $\lambda$ of this equation will be denoted by $\lambda_n$. 

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Thus we have shown that the problem
\[
\max_{a \in \mathbb{R}^m} a^T B_n a
\]
subject to
\[
a^T A_n a = 1
\]
is equivalent to determining
\[
\lambda_n,
\]
the solutions of
\[
\det(B_n - \lambda_n A_n) = 0, \quad n \in \mathbb{R}^N.
\]
The conclusion from this step is that, \( \varphi \) is rank-one convex if and only if
\[
1 \geq \max_{n \in \mathbb{R}^N} \lambda_n.
\]

Step 5. Observe that
\[
1 \geq \max_{a \in \mathbb{R}^m, n \in \mathbb{R}^N} n \odot a B a \odot n = \max_{n \in \mathbb{R}^N} \lambda_n \geq \sup_{j \in \mathbb{Z}^N} \lambda_j,
\]
because the solutions \( \lambda_j \) of
\[
\det(\tilde{B}_j - \lambda_j \tilde{A}_j) = 0, \quad j \in \mathbb{Z}^N,
\]
and the solutions \( \lambda_n \) of
\[
\det(B_n - \lambda_n A_n) = 0, \quad n \in \mathbb{R}^N
\]
are the same (they only differ in multiplicity).

Consequently, a quadratic \( \varphi \) is quasiconvex if and only if it is rank-one convex.

\section{Quasiconvexity for 4th degree homogeneous polynomials}

Suppose now that \( u \in C_c^\infty(Q, \mathbb{R}^m) \) where, for convenience, \( Q = (-\pi, \pi)^N \). Let \( \varphi_1, \varphi_2 : \mathbb{M}^{m \times N} \rightarrow \mathbb{R} \) be homogeneous polynomials of degree four, with \( \varphi_1 \) strictly convex and take
\[
\varphi(\xi) = \varphi_1(\xi) - c \varphi_2(\xi), \quad c \in \mathbb{R}.
\]
In order to determine the values of \( c \) for which the corresponding \( \varphi \) is quasiconvex, we want to determine the extrema of the function
\[
\frac{\int_Q \varphi_2(\xi + \nabla u(x)) - \varphi_2(\xi) \, dx}{\int_Q \varphi_1(\xi + \nabla u(x)) - \varphi_1(\xi) \, dx},
\]
for every \( \xi \in \mathbb{M}^{m \times N} \) and for every \( u \in C_c^\infty(Q, \mathbb{R}^m) \). In the case of \( \xi = 0 \), this will be answered by Theorem 2. For checking the quasiconvexity at the origin, the above quotient is much simpler
\[
\frac{\int_Q \varphi_2(\nabla u(x)) \, dx}{\int_Q \varphi_1(\nabla u(x)) \, dx},
\]
as $\varphi_i(0) = 0$. As the above quotient is the quotient of two homogeneous expressions of degree 4 in $\nabla u$, we can consider the equivalent problem of computing the extrema of

$$
\int_Q \varphi_2(\nabla u(x)) \, dx,
$$

subject to

$$
\int_Q \varphi_1(\nabla u(x)) \, dx = 1.
$$

We can reduce this infinite-dimensional problem to a finite dimensional one, as we did in the quadratic case, by taking the Fourier expansion of $u$. Furthermore, we can expand $u$ as a Fourier series with imaginary coefficients, with the help of the following lemma.

**Lemma 1** $\varphi : \mathbb{M}^{m \times N} \to \mathbb{R}$ is quasiconvex if and only if

$$
\int_{(-\pi,\pi)^N} \varphi(\xi + \nabla u(x)) \, dx \geq \int_{(-\pi,\pi)^N} \varphi(\xi) \, dx \tag{2}
$$

for each $\xi \in \mathbb{M}^{m \times N}$ and $u \in C_c^\infty((\pi,\mathbb{R}^m)$ such that $u(-x) = -u(x)$, $x \in (-\pi,\pi)^N$.

**Proof.** We only need to prove the “if” part. Suppose, by hypothesis, that $\varphi$ verifies (2). We want to prove that $\varphi$ is quasiconvex. For this purpose, consider an arbitrary $\xi \in \mathbb{M}^{m \times N}$ and take for domain the set $\Omega = (0,\pi) \times (-\pi,\pi)^{N-1}$. So one must verify the inequality of the definition of quasiconvexity for every $u \in C_c^\infty(\Omega, \mathbb{R})$ such that $u(-x) = -u(x)$, $x \in (\pi,\mathbb{R}^m)$.

Define the “odd” extension of $u$ to $Q = (-\pi,\pi)^N$ by

$$
U(x) = \begin{cases} 
  u(x), & x \in \Omega \\
  0, & x \in \{0\} \times (-\pi,\pi)^{N-1} \\
  -u(-x), & x \in \Omega' := (-\pi,0) \times (-\pi,\pi)^{N-1}.
\end{cases}
$$

In particular, the following properties hold: $U \in C_c^\infty((\pi,\mathbb{R}^m)$ with $U(-x) = -U(x)$, $\nabla U(-x) = \nabla U(x)$, $x \in (-\pi,\pi)^N$. Then

$$
\begin{align*}
2 \int_\Omega \varphi(\xi + \nabla u(x)) \, dx &= \int_\Omega \varphi(\xi + \nabla U(x)) \, dx + \int_\Omega \varphi(\xi + \nabla U(-x)) \, dx = \\
&= \int_\Omega \varphi(\xi + \nabla U(x)) \, dx + \int_{\Omega'} \varphi(\xi + \nabla U(-y)) \, dy = \int_{(-\pi,\pi)^N} \varphi(\xi + \nabla U(x)) \, dx \\
&\geq \int_{(-\pi,\pi)^N} \varphi(\xi) \, dx = 2 \int_\Omega \varphi(\xi) \, dx.
\end{align*}
$$

By the above lemma one can take, without loss of generality, $u \in C_c^\infty(Q, \mathbb{R}^m)$ with $u(-x) = -u(x)$, $x \in Q$. Then

$$
u(x) = \sum_{j \in \mathbb{Z}^N} c_j e^{ij \cdot x}, \quad C_j = \frac{1}{(2\pi)^N} \int_Q u(x) e^{-ij \cdot x} \, dx,
$$
and so, in particular we have
\[ \nabla u(x) = \sum_{j \in \mathbb{Z}^N} iC_j \otimes j e^{ij \cdot x}, \]
with
\[ C_{-j} = -C_j, \quad j \in \mathbb{Z}^N. \]
Since \( C_j \) is purely imaginary, \( iC_j \) is real and so we take as variables \( c_j = iC_j \), which are real.

The problem is now to find the extrema (now in the \( c_j \)) of
\[ \int_Q \varphi_2 \left( \sum_{j \in \mathbb{Z}^N} c_j \otimes j e^{ij \cdot x} \right) dx \]
subject to
\[ \int_Q \varphi_1 \left( \sum_{j \in \mathbb{Z}^N} c_j \otimes j e^{ij \cdot x} \right) dx = 1. \]

If \( \lambda \) is a Lagrange multiplier and \( C = (c_j)_{j \in \mathbb{Z}^N} \), we write
\[ L(C, \lambda) = \int_Q \varphi_2 \left( \sum_{j \in \mathbb{Z}^N} c_j \otimes j e^{ij \cdot x} \right) dx + \]
\[ - \lambda \left( \int_Q \varphi_1 \left( \sum_{j \in \mathbb{Z}^N} c_j \otimes j e^{ij \cdot x} \right) dx - 1 \right). \]

In order to obtain the first-order necessary conditions one has to compute
\[ \frac{\partial}{\partial c_j} \int_Q \varphi_i \left( \sum_{k \in \mathbb{Z}^N} c_k \otimes k e^{ik \cdot x} \right) dx = \]
\[ = \int_Q \nabla \varphi_i \left( \sum_{k \in \mathbb{Z}^N} c_k \otimes k e^{ik \cdot x} \right) \frac{\partial}{\partial c_j} \left( \sum_{k \in \mathbb{Z}^N} c_k \otimes k e^{ik \cdot x} \right) dx = \]
\[ = \int_Q \nabla \varphi_i \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{ik \cdot x} \right) (0, \ldots, 1, \ldots, 0) \otimes j (e^{ij \cdot x} + e^{-ij \cdot x}) dx, \]
where \( p = 1, \ldots, m, \ c_j = (c_j^1, \ldots, c_j^m) \) and \( (0, \ldots, 1, \ldots, 0) \) above means that all the coordinates are zero except the \( p \)-th one.

Since \( \varphi_i \) are homogeneous polynomials of degree four, one can write
\[ \varphi_1(\nabla u) = A_1(\nabla u, \nabla u, \nabla u, \nabla u), \quad \varphi_2(\nabla u) = A_2(\nabla u, \nabla u, \nabla u, \nabla u), \]
where \( A_1, A_2 \) are 4th order (totally symmetric) tensors with \( m^4 N^4 \) constant coefficients, and so the last equality above becomes
\[ 4 \int_Q A_1 \left( \sum_{k \in \mathbb{Z}^N} c_k \otimes k e^{ik \cdot x}, \sum_{l \in \mathbb{Z}^N} c_l \otimes l e^{il \cdot x}, \sum_{m \in \mathbb{Z}^N} c_m \otimes m e^{im \cdot x} \right) \]
\[ (0, \ldots, 1, \ldots, 0) \otimes j (e^{ij \cdot x} + e^{-ij \cdot x}) dx = \]
We are now interested in incorporating where

\[ Z = 4 \]

With respect to the last equation of the set of first-order necessary conditions, one obtains

\[
\int_Q e^{(k+l+m)x} (e^{ijx} + e^{-ijx}) \, dx =
\]

\[
= 4 \sum_{k,l,m \in \mathbb{Z}^N} A_1 (c_k \otimes k, c_l \otimes l, c_m \otimes m, (0, ..., 1, ..., 0) \otimes j)
\]

\[
\int_Q e^{(k+l+m)x} (e^{ijx} + e^{-ijx}) \, dx =
\]

\[
= 1 \text{ if } j + k + l + m = 0 \text{ or } j - k - l - m = 0
\]

\[
= 4 \sum_{k,l \in \mathbb{Z}^N} \left( A_1 ((0, ..., 1, ..., 0) \otimes j, c_k \otimes k, c_l \otimes l, c_{j+k+l} \otimes (j + k + l)) + A_1 ((0, ..., 1, ..., 0) \otimes j, c_k \otimes k, c_l \otimes l, c_{j-k-l} \otimes (j - k - l)) \right),
\]

where \( p = 1, ..., m \).

With respect to the last equation of the set of first-order necessary conditions, one obtains

\[
\int_Q \varphi_1 \left( \sum_{k \in \mathbb{Z}^N} c_k \otimes k e^{ikx} \right) \, dx - 1 =
\]

\[
= \int_Q \sum_{j,k,l,m \in \mathbb{Z}^N} A_1 (c_j \otimes j, c_k \otimes k, c_l \otimes l, c_m \otimes m) e^{(j+k+l+m)x} \, dx - 1 =
\]

\[
= \sum_{j,k,l \in \mathbb{Z}^N} A_1 (c_j \otimes j, c_k \otimes k, c_l \otimes l, c_{j+k+l} \otimes (j + k + l)) - 1 = 0.
\]

We are now interested in incorporating \( j, k, l, j + k + l \) inside the matrices \( A_1 \) and \( A_2 \), before writing the optimality conditions. That can be done using a procedure similar to the one used in the quadratic case, but using the formulas twice, as here we deal with fourth order tensors, instead of second order tensors. For a fourth order tensor

\[
A = \begin{pmatrix}
A_{1,1}^{1,1} & A_{1,2}^{1,1} & \cdots & A_{1,mN}^{1,1} & A_{2,1}^{1,1} & \cdots & A_{mN,mN}^{1,1} \\
A_{1,1}^{1,2} & A_{1,2}^{1,2} & \cdots & A_{1,mN}^{1,2} & A_{2,1}^{1,2} & \cdots & A_{mN,mN}^{1,2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{1,1}^{mN,mN} & A_{1,2}^{mN,mN} & \cdots & A_{1,mN}^{mN,mN} & A_{2,1}^{mN,mN} & \cdots & A_{mN,mN}^{mN,mN} \\
A_{2,1}^{1,1} & A_{2,2}^{1,1} & \cdots & A_{2,mN}^{1,1} & A_{2,1}^{1,2} & \cdots & A_{2,mN,mN}^{1,2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
A_{mN,mN}^{1,1} & A_{mN,mN}^{2,1} & \cdots & A_{mN,mN}^{1,mN} & A_{mN,mN}^{2,1} & \cdots & A_{mN,mN}^{mN,mN}
\end{pmatrix}.
\]
we put
\[
A(c_j \otimes j, c_k \otimes k, c_l \otimes l, c_{j+k+l} \otimes (j+k+l)) = \\
= l \otimes c_l \\
\begin{pmatrix}
j \otimes c_j A_1 c_k \otimes k & \ldots & j \otimes c_j A_{mO} c_k \otimes k \\
\vdots & \ddots & \vdots \\
j \otimes c_j A_{mO} c_k \otimes k & \ldots & j \otimes c_j A_{mO} c_k \otimes k
\end{pmatrix} c_{j+k+l} \otimes (j+k+l),
\]

\[A^a_b = (A^a_{b,d})_{d=1,\ldots,m}, \quad a, b = 1, \ldots, m.\]

First we deal with \( j \otimes c_j A^a_b c_k \otimes k. \)

For \( c_j = (c^1_j, \ldots, c^m_j), \quad j = (j_1, \ldots, j_N), \)

\[c_j \otimes j = \begin{pmatrix}
c^1_j j_1 \\
\vdots \\
c^m_j j_N
\end{pmatrix}.\]

Then we have \( j \otimes c_j A^a_b c_k \otimes k = c_j^T A^a_b (j, k) c_k, \)

where
\[
A^a_b (j, k) = \begin{pmatrix}
\alpha^a_{b,1} (j, k) & \ldots & \alpha^a_{b,m} (j, k) \\
\vdots & \ddots & \vdots \\
\alpha^a_{b,1} (j, k) & \ldots & \alpha^a_{b,m} (j, k)
\end{pmatrix},
\]

\[
\alpha^a_{b, p}(j, k) = \sum_{r=1}^{N} j_r k_r A^a_{b,r} A^a_{b,(p-1)N} + \sum_{r,s=1}^{N} (j_r k_s + j_s k_r) A^a_{b,s} A^a_{b,(p-1)N}, \quad p = 1, \ldots, m, \tag{3}
\]

\[
\alpha^a_{b, q}(j, k) = \sum_{r=1}^{N} j_r k_r A^a_{b,r} A^a_{b,(q-1)N} + \sum_{r,s=1}^{N} (j_r k_s + j_s k_r) A^a_{b,s} A^a_{b,(q-1)N}, \quad q < p, \quad q, p = 1, \ldots, m, \tag{4}
\]

\[
\alpha^a_{b, q}(j, k) = \sum_{r=1}^{N} j_r k_r A^a_{b,r} A^a_{b,(q-1)N} + \sum_{r,s=1}^{N} (j_r k_s + j_s k_r) A^a_{b,s} A^a_{b,(q-1)N}, \quad q > p, \quad q, p = 1, \ldots, m. \tag{5}
\]
So we have

\[
A \left( c_j \otimes j, c_k \otimes k, c_l \otimes l, c_{j+k+l} \otimes (j+k+l) \right) = \\
= l \otimes c_l \\
= c_l^T \begin{pmatrix}
  c_l^T A^1_{1}(j, k) c_k & ... & c_l^T A^1_{n}(j, k) c_k \\
  ... & ... & ... \\
  c_l^T A^m_{1}(j, k) c_k & ... & c_l^T A^m_{n}(j, k) c_k \\
\end{pmatrix} c_{j+k+l} \otimes (j+k+l) = \\
= c_l^T \begin{pmatrix}
  c_l^T A^1_{1}(j, k, l, j+k+l) c_k & ... & c_l^T A^1_{m}(j, k, l, j+k+l) c_k \\
  ... & ... & ... \\
  c_l^T A^m_{1}(j, k, l, j+k+l) c_k & ... & c_l^T A^m_{m}(j, k, l, j+k+l) c_k \\
\end{pmatrix}
\]

where

\[
A^p_{r}(j, k, l, j+k+l) = \sum_{r=1}^{N} l_r (j_r + k_r + l_r) A^r_{r+(p-1)N}(j, k) + \\
+ \sum_{r,s=1, r < s}^{N} l_r (j_r + k_s + l_s) A^r_{r+(p-1)N}(j, k) + l_s (j_r + k_r + l_r) A^s_{s+(p-1)N}(j, k),
\]

\( p = 1, ..., m, \quad q < p, \quad q, p = 1, ..., m, \)

\[
A^q_{r}(j, k, l, j+k+l) = \sum_{r=1}^{N} l_r (j_r + k_r + l_r) A^{r+(q-1)N}_{r+(p-1)N}(j, k) + \\
+ \sum_{r,s=1, r < s}^{N} l_r (j_r + k_s + l_s) A^{r+(q-1)N}_{r+(p-1)N}(j, k) + l_s (j_r + k_r + l_r) A^{s+(q-1)N}_{s+(p-1)N}(j, k),
\]

\( q > p, \quad q, p = 1, ..., m. \)

To achieve this formulas one just have to make a second iteration with the formulas (3), (4), (5) and use the distributivity of the matrix multiplication with respect to its sum. Finally, we have

\[
A \left( c_j \otimes j, c_k \otimes k, c_l \otimes l, c_{j+k+l} \otimes (j+k+l) \right) = A^{j+k,l,j+k+l}_{j+k,l,j+k+l}(c_j, c_k, c_l, c_{j+k+l}),
\]

for

\[
A^{j,k,l,j+k+l}_{j+k,l,j+k+l} = \begin{pmatrix}
  A^1_{1}(j, k, l, j+k+l) & ... & A^1_{m}(j, k, l, j+k+l) \\
  ... & ... & ... \\
  A^m_{1}(j, k, l, j+k+l) & ... & A^m_{m}(j, k, l, j+k+l)
\end{pmatrix}.
\]
Theorem 2

Let \(\phi\) be quasiconvex at zero if and only if

\[
\phi(\xi) = \phi_1(\xi) - c\phi_2(\xi).
\]

Then \(\phi\) is quasiconvex at zero if and only if

If we apply this method to \(A_1\) and \(A_2\) and simplify the notation by introducing

\[
a_{j,k,l,j+k+l} = A_{1,j,k,l,j+k+l},
\]
\[
b_{j,k,l,j+k+l} = A_{2,j,k,l,j+k+l},
\]

the first-order necessary conditions will then be

\[
\sum_{k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l})(1,0,...0) (c_k,c_l,c_{j+k+l}) + \\
+ (b_{j,k,l,j-k-l} - \lambda a_{j,k,l,j-k-l})(0,...,1,c_k,c_l,c_{j-k-l}) = 0,
\]

for each \(j \in \mathbb{Z}^N\)

\[
\sum_{k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l})(0,...,0,1) (c_k,c_l,c_{j+k+l}) + \\
+ (b_{j,k,l,j-k-l} - \lambda a_{j,k,l,j-k-l})(0,...,0,1,c_k,c_l,c_{j-k-l}) = 0,
\]

\[
\sum_{j,k,l \in \mathbb{Z}^N} a_{j,k,l,j+k+l}(c_j,c_k,c_l,c_{j+k+l}) = 1
\]

If \(C = (c_j)_{j \in \mathbb{Z}^N}\) is a critical point, we can multiply

\[
\sum_{k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l})(0,...,1,...0) (c_k,c_l,c_{j+k+l}) + \\
+ (b_{j,k,l,j-k-l} - \lambda a_{j,k,l,j-k-l})(0,...,1,...0,c_k,c_l,c_{j-k-l}) = 0
\]

by \(c^p\) and sum in \(p = 1,...,m\), thus obtaining

\[
\sum_{k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l})(c_j,c_k,c_l,c_{j+k+l}) + \\
+ (b_{j,k,l,j-k-l} - \lambda a_{j,k,l,j-k-l})(c_j,c_k,c_l,c_{j-k-l}) = 0.
\]

Then summing in \(j \in \mathbb{Z}^N\) gives

\[
2 \sum_{j,k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l}) c_j c_k c_{j+k+l} = 0,
\]

and using the last equation from (6) leads to

\[
\sum_{j,k,l \in \mathbb{Z}^N} b_{j,k,l,j+k+l}(c_j,c_k,c_l,c_{j+k+l}) = \lambda
\]

and then to

\[
\int_Q \varphi_2 \left( \sum_{k \in \mathbb{Z}^N} c_k \otimes k e^{ik \cdot x} \right) dx = \lambda.
\]

We are now entitled to formulate the following

**Theorem 2** Let \(\varphi_1, \varphi_2 : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}\) be homogeneous polynomials of degree four, with \(\varphi_1\) strictly convex and consider

\[
\varphi(\xi) = \varphi_1(\xi) - c \varphi_2(\xi).
\]

Then \(\varphi\) is quasiconvex at zero if and only if
1. \( c \in [c_-, c_+] \), if \( \frac{1}{c_+} - \frac{1}{c_-} < 0 \);
2. \( c \in (-\infty, c_+] \), if \( \frac{1}{c_-} = 0 \);
3. \( c \in [c_-, +\infty) \), if \( \frac{1}{c_+} = 0 \),

with
\[
\frac{1}{c_-} \left( \text{resp} \frac{1}{c_+} \right) = \inf \lambda (\text{resp} \sup \lambda),
\]
where the values of \( \lambda \) can be obtained as the solutions of (6).

Unfortunately, the set of equations (6) does not provide us with better understanding of the problem than those give by itself, even if we consider deformations with just a few terms. This is the major reason why we proceed to the case of the second gradients, where the \( c_j \) are scalars.

### 4 The case of the second gradients

Suppose that \( u \in C^\infty_c(Q, \mathbb{R}) \) where we set \( Q = (-\pi, \pi)^N \). Let \( \varphi_1, \varphi_2 : M_{sym}^{N \times N} \to \mathbb{R} \) be homogeneous polynomials of degree four, with \( \varphi_1 \) strictly convex and take
\[
\varphi(\xi) = \varphi_1(\xi) - \varphi_2(\xi), \ c \in \mathbb{R}.
\]

In order to determine the values of \( c \) for which the corresponding \( \varphi \) is 2-quasiconvex, we want to determine the extrema of the function
\[
\frac{\int_Q \varphi_2(\xi + \nabla^2 u(x)) - \varphi_2(\xi) \, dx}{\int_Q \varphi_1(\xi + \nabla^2 u(x)) - \varphi_1(\xi) \, dx},
\]
for every \( \xi \in M_{sym}^{N \times N} \) and for every \( u \in C^\infty_c(Q, \mathbb{R}) \). We will study the case \( \xi = 0 \), as in the gradient case. For checking the 2-quasiconvexity of \( \varphi \) at the origin, one must check
\[
\frac{\int_Q \varphi_2(\nabla^2 u(x)) \, dx}{\int_Q \varphi_1(\nabla^2 u(x)) \, dx}
\]
as \( \varphi_i(0) = 0 \). We can consider the equivalent problem of computing the extrema of
\[
\int_Q \varphi_2(\nabla^2 u(x)) \, dx,
\]
subject to
\[
\int_Q \varphi_1(\nabla^2 u(x)) \, dx = 1.
\]

We can now expand \( u \) as a Fourier series with real coefficients, with the help of the following lemma, whose proof is similar to the gradient case.

**Lemma 2** \( \varphi : M_{sym}^{N \times N} \to \mathbb{R} \) is 2-quasiconvex if and only if
\[
\int_{(-\pi, \pi)^N} \varphi(\xi + \nabla^2 u(x)) \, dx \geq \int_{(-\pi, \pi)^N} \varphi(\xi) \, dx \quad (7)
\]
for each \( \xi \in M_{sym}^{N \times N} \) and \( u \in C^\infty((-\pi, \pi)^N, \mathbb{R}) \) such that \( u(-x) = u(x), \ x \in (-\pi, \pi)^N \).
By the above lemma one can take, without loss of generality, \( u \in C_c^\infty(Q, \mathbb{R}) \) with \( u(-x) = u(x), \ x \in Q \). Then
\[
u(x) = \sum_{j \in \mathbb{Z}^N} c_j e^{ij.x}, \ c_j = \frac{1}{(2\pi)^N} \int_Q u(x) e^{-ij.x} dx,
\]
and so, in particular we have
\[
\nabla^2 u(x) = - \sum_{j \in \mathbb{Z}^N} j \otimes j c_j e^{ij.x},
\]
with
\[
\ c_{-j} = c_j, \ j \in \mathbb{Z}^N.
\]
The problem is now to find the extrema (now in the \( c_j \)'s) of
\[
\int_Q \varphi_2 \left( \sum_{j \in \mathbb{Z}^N} j \otimes j c_j e^{ij.x} \right) dx
\]
subject to
\[
\int_Q \varphi_1 \left( \sum_{j \in \mathbb{Z}^N} j \otimes j c_j e^{ij.x} \right) dx = 1.
\]
If \( \lambda \) is a Lagrange multiplier and \( C = (c_j)_{j \in \mathbb{Z}^N} \), we write
\[
L(C, \lambda) = \int_Q \varphi_2 \left( \sum_{j \in \mathbb{Z}^N} j \otimes j c_j e^{ij.x} \right) dx + \lambda \left( \int_Q \varphi_1 \left( \sum_{j \in \mathbb{Z}^N} j \otimes j c_j e^{ij.x} \right) dx - 1 \right).
\]
In order to obtain the first-order necessary conditions one has to compute
\[
\frac{\partial}{\partial c_j} \int_Q \varphi_1 \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{ik.x} \right) dx = \int_Q \nabla \varphi_1 \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{ik.x} \right) \frac{\partial}{\partial c_j} \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{ik.x} \right) dx
\]
\[
= \int_Q \nabla \varphi_1 \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{ik.x} \right) j \otimes j (e^{ij.x} + e^{-ij.x}) dx.
\]
Because \( \varphi_1 \) are homogeneous polynomials of degree four, we can write
\[
\varphi_1(\nabla^2 u) = A_1(\nabla^2 u, \nabla^2 u, \nabla^2 u, \nabla^2 u), \ \varphi_2(\nabla^2 u) = A_2(\nabla^2 u, \nabla^2 u, \nabla^2 u, \nabla^2 u),
\]
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where $A_1, A_2$ are 4th order (totally symmetric) tensors with $N^8$ constant coefficients, and so the last equality above becomes

$$\begin{align*}
4 \int_Q A_i \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{ik.x}, \sum_{l \in \mathbb{Z}^N} l \otimes l c_l e^{il.x}, \sum_{m \in \mathbb{Z}^N} m \otimes m c_m e^{im.x} \right) j \otimes j (e^{ij.x} + e^{-ij.x}) \, dx = \\
= 4 \int_Q \sum_{k,l,m \in \mathbb{Z}^N} A_i (k \otimes k, l \otimes l, m \otimes m) c_k c_l c_m e^{i(k+l+m).x} j \otimes j (e^{ij.x} + e^{-ij.x}) \, dx = \\
= 4 \sum_{k,l,m \in \mathbb{Z}^N} A_i (k \otimes k, l \otimes l, m \otimes m, j \otimes j) c_k c_l c_m \\
\int_Q e^{i(k+l+m).x}(e^{ij.x} + e^{-ij.x}) \, dx = 0 \quad \text{if } j + k + l = 0 \text{ or } j - k - l = 0
\end{align*}$$

$$\begin{align*}
= 4 \sum_{k,l \in \mathbb{Z}^N} (A_i (j \otimes j, k \otimes k, l \otimes l, (j + k + l) \otimes (j + k + l)) c_k c_l c_j c_{j+k+l} + \\
+ A_i (j \otimes j, k \otimes k, l \otimes l, (j - k - l) \otimes (j - k - l)) c_k c_l c_{j-k-l})
\end{align*}$$

With respect to the last equation of the set of first-order necessary conditions, one obtains

$$\begin{align*}
\int_Q \varphi_1 \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{ik.x} \right) \, dx - 1 = \\
= \int_Q \sum_{j,k,l \in \mathbb{Z}^N} A_i (j \otimes j, k \otimes k, l \otimes l, m \otimes m) c_j c_k c_l c_m e^{i(j+k+l+m).x} \, dx - 1 = \\
= \sum_{j,k,l \in \mathbb{Z}^N} A_i (j \otimes j, k \otimes k, l \otimes l, (j + k + l) \otimes (j + k + l)) c_j c_k c_l c_{j+k+l} - 1 = 0
\end{align*}$$

Consequently, simplifying the notation and writing

$$\begin{align*}
a_{j,k,l,j+k+l} &= A_i (j \otimes j, k \otimes k, l \otimes l, (j + k + l) \otimes (j + k + l)), \\
b_{j,k,l,j+k+l} &= A_i (j \otimes j, k \otimes k, l \otimes l, (j + k + l) \otimes (j + k + l)),
\end{align*}$$

the desired set of equations will read

$$\begin{align*}
\left\{ \begin{array}{ll}
\sum_{k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l}) c_k c_l c_{j+k+l} + \\
+ (b_{j,k,l,j-k-l} - \lambda a_{j,k,l,j-k-l}) c_k c_l c_{j-k-l} \right. = 0, \quad \text{for each } j \in \mathbb{Z}^N, \\
\sum_{j,k,l \in \mathbb{Z}^N} a_{j,k,l,j+k+l} c_j c_k c_l c_{j+k+l} = 1.
\end{array} \right.
\end{align*}$$
If $C = (c_j)_{j \in \mathbb{Z}^N}$ is a critical point, we can multiply

$$
\sum_{k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l}) c_k c_l c_{j+k+l} + (b_{j,k,l,j-k-l} - \lambda a_{j,k,l,j-k-l}) c_k c_l c_{j-k-l} = 0
$$

by $c_j$ and sum in $j \in \mathbb{Z}^N$, thus obtaining

$$
2 \sum_{j,k,l \in \mathbb{Z}^N} (b_{j,k,l,j+k+l} - \lambda a_{j,k,l,j+k+l}) c_j c_k c_l c_{j+k+l} = 0.
$$

Then using the last equation from (8) leads to

$$
\sum_{j,k,l \in \mathbb{Z}^N} b_{j,k,l,j+k+l} c_j c_k c_l c_{j+k+l} = \lambda
$$

and then to

$$
\int_Q \varphi_2 \left( \sum_{k \in \mathbb{Z}^N} k \otimes k c_k e^{i k \cdot x} \right) dx = \lambda.
$$

We are now entitled to formulate the following

**Theorem 3** Let $\varphi_1, \varphi_2 : \mathbb{R}^{N \times N} \to \mathbb{R}$ be homogeneous polynomials of degree four, with $\varphi_1$ strictly convex and consider

$$
\varphi(\xi) = \varphi_1(\xi) - c \varphi_2(\xi).
$$

Then $\varphi$ is 2-quasiconvex at zero if and only if

1. $c \in [c_-, c_+]$, if $\frac{1}{c_+} - \frac{1}{c_-} < 0$;
2. $c \in (-\infty, c_+]$, if $\frac{1}{c_-} = 0$;
3. $c \in [c_-, +\infty)$, if $\frac{1}{c_+} = 0$,

with

$$
\frac{1}{c_-} \left( \text{resp} \frac{1}{c_+} \right) = \inf \lambda (\text{resp} \sup \lambda),
$$

where the values of $\lambda$ can be obtained as the solutions of (8).

In general it is hard to solve (8), because the equations are extremely connected and determined as they share its variables, and so it is not possible to simplify the problem as one can do in the quadratic case. Nevertheless, one can consider, in some particular cases, Fourier expansions of $u$ with just a few terms, aiming to understand better the details involved.

### 5 The classical examples for N=2

In this case $j = (j_1, j_2) \in \mathbb{Z}^2$ and we will consider competing deformations with just a few terms.
5.1 One term

In this case we consider \( c_j = c_{-j} \neq 0, \ c_k = 0 \) for \( k \neq j, -j \). Notice that (8) is now

\[
\begin{aligned}
6(b_{j,j,j,j} - \lambda a_{j,j,j,j}) c_j^3 &= 0, \\
6 a_{j,j,j,j} c_j^4 &= 1.
\end{aligned}
\]

For

\[
\varphi(\xi) = \varphi_1(\xi) - \varphi_2(\xi) = |\xi|^4 - |\xi|^2 \det \xi
\]

and

\[
\varphi(\xi) = \varphi_1(\xi) - \varphi_2(\xi) = |\xi|^4 - (\det \xi)^2,
\]

one has

\[
a_{j,j,j,j} = (j_1^2 + j_2^2)^4,
\]

\[
b_{j,j,j,j} = 0,
\]

and then \( \lambda = 0 \). Consequently, this case is not interesting.

5.2 Two terms

For this case we consider \( c_j, c_k \neq 0, \ c_{-j} = c_j, \ c_{-k} = c_k \) (with \( k \neq \alpha j \), otherwise it will lead to \( \lambda = 0 \)), \( c_l = 0 \) for \( l \neq j, -j, k, -k \). The first order necessary conditions will now be

\[
\begin{aligned}
6(b_{j,j,j,j} - \lambda a_{j,j,j,j}) c_j^3 + 12(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_j c_k^2 &= 0, \\
12(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_j^2 c_k + 6(b_{k,k,k,k} - \lambda a_{k,k,k,k}) c_k^3 &= 0 \\
6 a_{j,j,j,j} c_j^4 + 24 a_{j,j,j,j} c_j^2 c_k^2 + 6 a_{k,k,k,k} c_k^4 &= 1
\end{aligned}
\]

\[
\begin{aligned}
(b_{j,j,j,j} - \lambda a_{j,j,j,j}) c_j^2 + 2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_k^2 &= 0, \\
2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_j^2 + (b_{k,k,k,k} - \lambda a_{k,k,k,k}) c_k^2 &= 0 \\
6 a_{j,j,j,j} c_j^4 + 24 a_{j,j,j,j} c_j^2 c_k^2 + 6 a_{k,k,k,k} c_k^4 &= 1
\end{aligned}
\]

In the case of

\[
\varphi_2(\xi) = (\det \xi)^2 \quad \text{or} \quad \varphi_2(\xi) = |\xi|^2 \det \xi,
\]

this system is equivalent to

\[
\begin{aligned}
-\lambda a_{j,j,j,j} c_j^2 + 2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_k^2 &= 0, \\
2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_j^2 - \lambda a_{k,k,k,k} c_k^2 &= 0 \\
6 a_{j,j,j,j} c_j^4 + 24 a_{j,j,j,j} c_j^2 c_k^2 + 6 a_{k,k,k,k} c_k^4 &= 1.
\end{aligned}
\]

Notice that (for \( \varphi_1 = |\xi|^4 \))

\[
a_{j,j,k,k} = \frac{1}{3}(j_1^2 + j_2^2)(k_1^2 + k_2^2)^2 + \frac{2}{3}(j_1 k_1 + j_2 k_2)^4
\]

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\[
\begin{align*}
&b_{j,j,k,k} = \frac{1}{3} (j_1 k_1 + j_2 k_2)^2 (j_2 k_1 - j_1 k_2)^2, \quad \varphi_2(\xi) = |\xi|^2 \det \xi \\
&b_{j,j,k,k} = \frac{1}{6} (j_1 k_2 - j_2 k_1)^4, \quad \varphi_2(\xi) = (\det \xi)^2
\end{align*}
\]
are non-negative, and so \( \lambda \) must be non-negative also, otherwise the above system is impossible.

Furthermore, we have to impose

\[
\lambda^2 a_{j,j,j,j} a_{k,k,k,k} + 4 (b_{j,j,k,k} - \lambda a_{j,j,k,k})^2 = 0 \iff \lambda = \frac{2b_{j,j,k,k}}{\sqrt{a_{j,j,j,j} a_{k,k,k,k} + 2a_{j,j,k,k}}},
\]
again by the non-negativity of \( \lambda \). To compute the extrema of \( \lambda \), one has only to compute the maximum because \( \lambda \geq 0 \), and 0 is attained.

With \( \varphi_2(\xi) = (\det \xi)^2 \), one has

\[
\sup_{j,k} \lambda = \sup_{j,k} \frac{(j_1 k_2 - j_2 k_1)^4}{5(j_1^2 + j_2^2)^2(k_1^2 + k_2^2)^2 + 4(j_1 k_1 + j_2 k_2)^2}.
\]
In the above quotient, \( j, k \) can be taken such that \( |j| = |k| = 1 \), leading to

\[
\frac{(j_1 k_2 - j_2 k_1)^4}{5 + 4(j_1 k_1 + j_2 k_2)^2},
\]
and then

\[
\sup_{j,k} \lambda = \sup_{j,k} \frac{(j_1 k_2 - j_2 k_1)^4}{5 + 4(j_1 k_1 + j_2 k_2)^2} = \frac{1}{5}.
\]
This maximum is attained in the initial fraction by taking any 2 orthonormal vectors \( j, k \).

For

\[
\varphi_2(\xi) = |\xi|^2 \det \xi,
\]
one has to compute

\[
\sup_{j,k} \lambda = \sup_{j,k} \frac{2(j_1 k_1 + j_2 k_2)^2 (j_1 k_2 - j_2 k_1)^2}{5(j_1^2 + j_2^2)^2(k_1^2 + k_2^2)^2 + 4(j_1 k_1 + j_2 k_2)^2}.
\]
In this quotient, one can again take \( j = (j_1, j_2), k = (k_1, k_2) \) with \( |j| = |k| = 1 \) and then get

\[
\sup_{j,k} \lambda = \sup_{j,k} \frac{2(j, k)^2 (j, \tilde{k})^2}{5 + 4(j, k)^2},
\]
where \( \tilde{k} = (-k_2, k_1) \). As \( j, k \) are unit vectors and \( j, \tilde{k} \) are orthogonal, this can be further simplified into

\[
\max_{\theta \in [0, 2\pi)} \frac{2 \cos^2(\theta) (1 - \cos^2(\theta))}{5 + 4 \cos^2(\theta)} = \max_{x \in [0,1]} \frac{2x (1 - x)}{5 + 4x^2} = -\frac{1}{4} + \frac{3\sqrt{5}}{20}.
\]
As we easily observe, the quotient
\[
\frac{\int_Q \varphi_2(\nabla^2 u(x)) \, dx}{\int_Q \varphi_1(\nabla^2 u(x)) \, dx}
\]
is obviously positive in the case were
\[
\varphi_1 = |\xi|^4, \quad \varphi_2 = (\det \xi)^2,
\]
but it surely takes both positive and negative values when
\[
\varphi_2 = |\xi|^2 \det \xi.
\]

One might be tempted to try to find an counterexample for \(N = 2\), but notice that we are restricted to the case were the first moment is 0. First of all, we need to know for which values of \(c \in \mathbb{R}\) the corresponding \(\varphi\) is convex along its characteristic cone. Then, if the smallest value obtained is zero, this will provide the desired conclusion (if one has an example of a periodic deformation with more than two terms, which is easy).

The characteristic cone ([12]) associated with 2-quasiconvexity is
\[
\Lambda = \{a \otimes a, \ a \in \mathbb{R}^N\}.
\]
The determination of which values of \(c\) provide functions \(\varphi\) that are convex along the directions of \(\Lambda\) can be done with the techniques developed in [4], applied to this particular case. It is easy to conclude that
\[
\varphi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}
\]
is convex along \(\Lambda\) if and only if
\[
c \in \left[\begin{array}{cc}
-\frac{4}{\sqrt{3}} & \frac{4}{\sqrt{3}} \\
\frac{4}{3} & \frac{4}{3}
\end{array}\right],
\]
that is the maximum value for \(\lambda\) (with at most two terms, considering all possible first moments) is \(\frac{4}{\sqrt{3}}\) and the minimum is \(-\frac{4}{\sqrt{3}}\). It could seem quite surprising that the values are exactly the same here, but in fact the computations in [4] for the classical example of [10] shows that the extrema are attained, e.g., for a first moment
\[
\xi = \left(\begin{array}{cc}
\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{\sqrt{2}}
\end{array}\right),
\]
which is, in particular, a symmetric matrix, and for any direction, provided that it is of rank-one (that is, in particular one can choose to take a matrix in \(\Lambda\)).

We have then to proceed to computations with deformations of three or more terms, if we want to study the possibility of finding a counterexample.
5.3 Three terms

In this case we consider three non-zero terms \( c_j, c_k, c_l \neq 0 \). As \( N = 2 \), we now that one of the \( j, k, l \) must be a linear combination of the other two. We must analyze several subcases. In all the subcases we get \( \lambda \geq 0 \) from the necessary conditions (8), except one, which will be treated below. For example, in the subcase were \( j, k, l \), with \( j \neq \alpha k, k \neq \alpha l, l \neq \alpha j \), we have

\[
\begin{align*}
\left\{ \begin{array}{l}
-\lambda a_{j,j,j,j} c_j^4 + 2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_k^2 + 2(b_{j,j,l,l} - \lambda a_{j,j,l,l}) c_l^2 = 0 \\
2(b_{j,j,k,k} - \lambda a_{j,j,k,k}) c_k^2 - \lambda a_{k,k,k,k} c_k^2 + 2(b_{k,k,l,l} - \lambda a_{k,k,l,l}) c_l^2 = 0 \\
2(b_{j,j,l,l} - \lambda a_{j,j,l,l}) c_l^2 + 2(b_{k,k,l,l} - \lambda a_{k,k,l,l}) c_l^2 - \lambda a_{l,l,l,l} c_l^2 = 0 \\
6a_{j,j,j,j} c_j^4 + 24a_{j,j,k,k} c_j^2 c_k^2 + 24a_{j,j,l,l} c_j^2 c_l^2 + 6a_{k,k,k,k} c_k^4 + 6a_{k,k,l,l} c_k^2 c_l^2 + 6a_{l,l,l,l} c_l^4 = 1,
\end{array} \right.
\]

and in the other subcases we have at least one equation of the kind of the first 3 equations of this system, which implies that \( \lambda \geq 0 \), as stated (with the above mentioned exception). Despite the fact that this system looks harmless, the computations involved to solve it become too hard to find the exact solutions, as we did in the previous cases with less terms, and the same happens in the other subcases, in general.

The exception in terms of the positivity of \( \lambda \) is the subcase were \( l = 2j + k \) for \( k \neq \alpha j, \alpha \in \mathbb{R} \). In this subcase it is possible to obtain negative values of \( \lambda \) (for \( \varphi_2(\xi) = |\xi|^2 \det \xi \)), which means that for deformations with three terms, the quotient

\[
\frac{\int_Q \varphi_2(\nabla^2 u(x)) \, dx}{\int_Q \varphi_1(\nabla^2 u(x)) \, dx}
\]

attain negative values. An example of such a deformation is

\[
u(x_1, x_2) = -\frac{1}{2} \left( e^{i(1,0) \cdot (x_1, x_2)} + e^{-i(1,0) \cdot (x_1, x_2)} \right) + \\
-\frac{1}{5} \left( e^{i(0,1) \cdot (x_1, x_2)} + e^{-i(0,1) \cdot (x_1, x_2)} \right) + \frac{1}{20} \left( e^{i(2,1) \cdot (x_1, x_2)} + e^{-i(2,1) \cdot (x_1, x_2)} \right),
\]

and the corresponding value attained,

\[
\lambda = -\frac{2944}{61971}.
\]

We recall that this does not provide any counterexample, because the minimum value attained by \( \lambda \) with at most two terms (if the first moment is not fixed) is

\[
\lambda_{\min} = -\frac{\sqrt{3}}{4}.
\]

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References


