Some questions concerning geometric inverse problems

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Resumen

We present some recent results and open questions concerning the geometric inverse problem of the identification of a rigid structure. Several equations are considered, all of them connected to specific applications: a linear elliptic equation, some stationary elasticity systems and also the wave equation. We discuss uniqueness and reconstruction algorithms.

1. The case of the Laplace equation

1.1. Formulation. Uniqueness and reconstruction results

Let $\Omega \subset \mathbb{R}^N$ be a simply connected bounded open set ($N = 2$ or $N = 3$) whose boundary $\partial \Omega$ is of class $W^{1,\infty}$. Let $\gamma$ be a nonempty open subset of $\partial \Omega$ and let us denote by $1_\gamma$ the characteristic function of $\gamma$.

Let $D^*$ be a fixed nonempty open set such that $D^* \subset \subset \Omega$. We will consider the following family of subsets of $\Omega$:

$$\mathcal{D} = \{ D \subset \Omega : D \text{ is a simply connected nonempty open set, } \partial D \text{ is of class } W^{1,\infty}, D \subset \subset D^* \}.$$  

We will deal with the following inverse problem:

Given $\varphi$ in an appropriate space, find a set $D \in \mathcal{D}$ such that a solution $u$ of the Laplace equation

$$\begin{cases}
-\Delta u = 0 & \text{in } \Omega \setminus \overline{D} \\
u = \varphi & \text{on } \partial \Omega \\
u = 0 & \text{on } \partial D
\end{cases}$$

(1)
satisfies the additional condition
\[ \frac{\partial u}{\partial n} = \alpha \quad \text{on} \quad \gamma. \tag{2} \]

Let us first examine uniqueness for this problem. Let \( D^0 \) and \( D^1 \) be two sets in \( D \) and assume that
\[
\begin{aligned}
-\Delta u^i &= 0 \quad \text{in} \quad \Omega \setminus \overline{D^i} \\
u^i &= \varphi \quad \text{on} \quad \partial \Omega \\
u^i &= 0 \quad \text{on} \quad \partial D^i
\end{aligned}
\tag{3}
\]
for \( i = 0 \) and \( i = 1 \). We have the following result:

**Theorem 1.1** Assume that \( \varphi \neq 0 \). Let \( D^0 \) and \( D^1 \) be two sets in \( D \), let \( u^i \) be the unique solution of (3) and let us set \( \alpha^i = \frac{\partial u^i}{\partial n} \) for \( i = 0, 1 \). Then, if
\[ \alpha^0 = \alpha^1 \quad \text{on} \quad \gamma, \tag{4} \]
one has \( D^0 = D^1 \).

Results of this kind are well known. The key point in their proofs is unique continuation, a property that is known to hold for the Laplace equation; for details, see for instance [3].

Now, let us assume that \( D \) known and let us consider a local reconstruction question for our inverse problem. In order to represent the deformations of a set \( D \in D \), let us introduce
\[
W = \{ m \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N) : \|m\|_{W^{2,\infty}} \leq \varepsilon, \quad m = 0 \quad \text{in} \quad \Omega \setminus D^* \},
\]
where \( \varepsilon > 0 \) is small enough. For each \( m \in W \), we define a new domain \( D + m \) with
\[
D + m = \{ z \in \mathbb{R}^N : z = x + m(x), \quad x \in D \}
\]
and we consider the “perturbed” system
\[
\begin{aligned}
-\Delta u^m &= 0 \quad \text{in} \quad \Omega \setminus \overline{(D + m)} \\
u^m &= \varphi \quad \text{on} \quad \partial \Omega \\
u^m &= 0 \quad \text{on} \quad \partial(D + m)
\end{aligned}
\tag{5}
\]
We then have the following result:

**Theorem 1.2** Assume that \( \varphi \in H^{3/2}(\partial \Omega) \), \( \varphi \neq 0 \), \( \alpha = \frac{\partial u}{\partial n} |_{\gamma} \) and \( \alpha^m = \frac{\partial u^m}{\partial n} |_{\gamma} \). Also, assume that \( (m \cdot n)|_{\partial D} \in M \), where \( M \) is a linear space satisfying \( M \subset W^{1,\infty}(\partial D) \), \( \dim M < \infty \). Then there exists a computable function \( G : L^2(\gamma) \rightarrow M \) such that
\[
(m \cdot n)|_{\partial D} = G(\alpha^m - \alpha) + o(m) \quad \text{for all small} \quad m.
\]

**Remark 1.3** The proof consists of two steps. First, we express \( \alpha^m - \alpha \), up to second order terms, as the normal derivative on \( \gamma \) of the solution of a linear problem; to this end, we use domain variation techniques, see [4]. Secondly, assuming that \( \dim M < +\infty \), we compute explicitly this normal derivative by solving a finite number of control problems; this concerns data assimilation techniques, see [6].

**Remark 1.4** It is also possible to state and analyze a finite-dimensional (approximated) version of the inverse problem (1)–(2). The details will be given in a forthcoming paper.
1.2. A penalized problem

Let us now consider the following penalized problem:

\[
\inf_{m \in \mathcal{M}_{ad}} \frac{1}{2} ||\alpha^m - \alpha||_{L^2(\gamma)}^2 + \frac{\varepsilon}{2} ||m||_M^2,
\]

(6)

where \( \mathcal{M}_{ad} \subset M \) is a closed convex set.

The associated optimality system is as follows:

\[
\begin{cases}
-\Delta u^\varepsilon = 0 & \text{in } \Omega \setminus (D + m^\varepsilon) \\
u^\varepsilon = \tilde{\phi} & \text{on } \partial \Omega \\
u^\varepsilon = 0 & \text{on } \partial(D + m^\varepsilon)
\end{cases}
\]

\[
\begin{cases}
-\Delta \varphi^\varepsilon = 0 \\
\varphi^\varepsilon = (\alpha^m - \alpha)1_\gamma \\
\varphi^\varepsilon = 0
\end{cases}
\]

and

\[
\int_{\partial(D + m^\varepsilon)} \left( -\frac{\partial u^\varepsilon}{\partial n} \right) \left( \frac{\partial \varphi^\varepsilon}{\partial n} \right) (p - m^\varepsilon) \cdot n + \varepsilon (m^\varepsilon, p - m^\varepsilon)_M \geq 0
\]

for all \( p \in \mathcal{M}_{ad}, \ m^\varepsilon \in \mathcal{M}_{ad} \).

It is not difficult to get the following estimates:

\[
||\alpha^m||_{L^2(\gamma)} \leq C, \quad ||\varphi^m||_{H^r(\gamma)} \leq C \quad \text{for small } r.
\]

Then we have an interesting, but nontrivial, open question:

For each \( \varepsilon > 0 \), let \( m^\varepsilon \) be the solution to (6). Then, do we have, under appropriate hypotheses, the convergence properties

\[
m^\varepsilon \to m^*, \quad \alpha^m \to \alpha^* \quad \text{with } \alpha^* = \alpha ?
\]

2. Stationary elasticity systems

2.1. The isotropic case

Let us consider now a similar geometrical inverse problem for the \( N \)-dimensional stationary Lamé system:

\[
\begin{cases}
-\nabla \cdot (\mu(x)(\nabla u + \nabla u^t)) - \nabla(\lambda(x)\nabla \cdot u) = 0 & \text{in } \Omega \setminus \overline{D} \\
u = \varphi & \text{on } \partial \Omega \\
u = 0 & \text{on } \partial D
\end{cases}
\]

(7)

where the coefficients \( \lambda \) and \( \mu \) are assumed to belong to \( L^\infty(\Omega) \) and satisfy the usual ellipticity assumptions. The observation is now

\[
\alpha = \mu(\nabla u + \nabla u^t) \cdot n + \lambda(\nabla \cdot u)n \quad \text{on } \gamma.
\]
We can prove uniqueness and partial identification results similar to those indicated in Section 1 for the Laplace equation. Again, the key point is the unique continuation property.

In the case $N = 2$, this property is fulfilled when $\lambda$ is Lipschitz-continuous and $\mu \in L^\infty(\Omega)$, see [2]. For $N \geq 3$, stronger hypotheses are needed. In particular, unique continuation holds if $\lambda, \mu \in C^{1,1}(\Omega)$, see [1].

2.2. The anisotropic case

This is more complicated. The system is

\[
\begin{align*}
\nabla \cdot \sigma(u) &= 0 \quad \text{in } \Omega \setminus D \\
u &= \varphi \quad \text{on } \partial \Omega \\
u &= 0 \quad \text{on } \partial D
\end{align*}
\]

The components of the stress tensor $\sigma(u)$ are given by

\[
\sigma_{kl}(u) = \sum_{i,j=1}^{3} a_{ijkl} \varepsilon_{ij}(u)
\]

with

\[
\varepsilon_{kl}(u) = \frac{1}{2} (\partial_k u_l + \partial_l u_k)
\]

for all $k, l = 1, 2, 3$. The observation is now

\[
\alpha = \sigma(u) \cdot n \quad \text{on } \gamma.
\]

Again, in order to get uniqueness and partial identification results (when $m \cdot n \in M$, $\dim M < +\infty$), we need a unique continuation property. When $N = 2$, this property is proved in [5] under the assumption

\[
a_{ijkl} \in W^{1,\infty}(\Omega) \quad \forall i, j, k, l.
\]

For $N \geq 3$, the conditions that must be imposed to the coefficients $a_{ijkl}$ to ensure unique continuation are unknown.

3. The one-dimensional wave equation

Now, assume that $\Omega = (0,1) \subset \mathbb{R}$ and $D = (a,b) \subset \Omega$. We will deal with the following inverse problem:

Given $\varphi_0, \varphi_1$ in appropriate spaces, find an interval $D = (a,b)$ such that the solution $u$ of the wave problem

\[
\begin{align*}
u_{tt} - u_{xx} &= 0 \quad (x,t) \in (\Omega \setminus D) \times (0,T) \\
u(0,t) &= \varphi_0(t), \quad u(1,t) = \varphi_1(t), \quad t \in (0,T) \\
u(a,t) &= 0, \quad u(b,t) = 0, \quad t \in (0,T) \\
u(x,0) &= 0, \quad u_t(x,0) = 0 \quad x \in (\Omega \setminus D
\end{align*}
\]

satisfies the additional conditions

\[
u_x(0,t) = \alpha(t), \quad u_x(1,t) = \beta(t).
\]
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Notice that this problem can be decomposed and rewritten as a system of two uncoupled (sub)problems, each of them at one side of \((a, b)\). For instance, to the left we have to solve the following:

Given \(\varphi_0\) in an appropriate space, find \(a\) such that a solution \(u\) of the wave problem

\[
\begin{aligned}
&u_{tt} - u_{xx} = 0 \quad (x,t) \in (0,a) \times (0,T) \\
u(0,t) = \varphi_0(t), &\quad u(a,t) = 0, \quad t \in (0,T) \\
u(x,0) = 0, &\quad u_t(x,0) = 0 \quad x \in (0,a)
\end{aligned}
\]  

satisfies the additional condition

\[u_x(0,t) = \alpha(t).\]  

3.1. Uniqueness

Let \(a_0\) and \(a_1\) be two solutions of \((11)–(12)\). Let us denote by \(u^i\) the solution to \((11)\) associated to \(a_i\).

**Theorem 3.1** Assume that \(T > 2\), \(\varphi_0 \in C^2([0, +\infty))\), \(\varphi_0(0) = \varphi'_0(0) = \varphi''_0(0) = 0\) and \(\varphi_0 \neq 0\). If \(u_{x}^0|_{x=0} = u_{x}^1|_{x=0} \) in \([0, T]\), then \(a_0 = a_1\).

Once more, the proof relies on a unique continuation property of \((11)\), that holds for all \(a \in (0,1)\) whenever \(T > 2\).

3.2. A reconstruction method

Again, let us assume that \(a \in (0,1)\) is known and let us try to find \(m\) from the observations corresponding to \(a\) and \(a + m\).

We introduce \(u^m\), with

\[
\begin{aligned}
u_{tt}^m - u_{xx}^m = 0 &\quad \text{in } (0,a + m) \times (0,T) \\
u^m(0,t) = \varphi_0(t), &\quad u^m(a + m,t) = 0 \quad \text{in } (0,T) \\
u(x,0) = 0, &\quad u_t(x,0) = 0 \quad \text{in } [0,a + m]
\end{aligned}
\]

We can obtain the second order variation formula

\[\alpha^m - \alpha = A(t) m + \frac{1}{2} B(t) m^2 + o(m^2),\]

where \(A(t) = z_x(0,t), B(t) = k_x(0,t)\) (the first and second derivatives),

\[
\begin{aligned}
z_{tt} - z_{xx} = 0 &\quad \text{in } (0,a) \times (0,T) \\z(0,t) = 0, &\quad z(a,t) = -u_x(a,t) \quad \text{in } (0,T) \\
z(x,0) = 0, &\quad z_t(x,0) = 0 \quad \text{in } (0,a)
\end{aligned}
\]

and

\[
\begin{aligned}
k_{tt} - k_{xx} = 0 &\quad \text{in } (0,a) \times (0,T) \\k(0,t) = 0, &\quad k(a,t) = -u_{xx}(a,t) \quad \text{in } (0,T) \\
k(x,0) = 0, &\quad k_t(x,0) = 0 \quad \text{in } (0,a)
\end{aligned}
\]
Therefore, it is completely natural to compute $m$ by solving the following extremal problem, where $\alpha$, $\tilde{\alpha}$, $A$ and $B$ are given:

$$\inf_{m \in R} ||(\tilde{\alpha} - \alpha) - A m - \frac{B}{2} m^2||^2.$$ 

Here, we can chose $\| \cdot \| = \| \cdot \|_{L^2(0,T)}$ or $\| \cdot \|_{H^1(0,T)}$. It is easy to see that the task is reduced to the search of the minimum of a polynomial of order 4.

**Remark 3.2** Similar questions can be considered for $N$-dimensional wave equations and nonstationary elasticity systems. Again, the key point is to understand under which conditions one has unique continuation. The detailed proofs, further extensions of the results in this work and some numerical experiments will appear in a forthcoming paper.

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**References**


