Resolución numérica de ecuaciones de convección-difusión con datos en L^1 por métodos distributivos

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Abstract

This paper deals with the analysis of piecewise finite element approximations of convection-diffusion equations with data in L^1. We discretize the convection operator by the PSI (Positive Streamwise Implicit) scheme, and the diffusion operator by the standard Galerkin method, using conforming $\mathbb{P}_1$ finite elements. We prove that this approximation satisfies the discrete maximum principle and converges to the unique renormalized solution in $W^{1,q}(\Omega)$, $1 \leq q < \frac{d}{d-1}$. We also prove some error estimates in this norm and include relevant numerical tests for data with low smoothness. Our error estimates appear to be roughly accurate, in view of the results.

1 Introduction

This paper deals with finite element approximations of convection-diffusion equations with data in L^1. We shall study the following problem:

$$\begin{cases} a \cdot \nabla u - \operatorname{div}(A \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^d$, $d \geq 2$, $a \in L^p(\Omega)^d$, $p > d$, is a given divergence-free velocity field,

$$\nabla \cdot a = 0, \tag{1.2}$$

$A \in L^\infty(\Omega)^{d\times d}$ is a uniformly positive definite and bounded matrix field.

$$M|\xi|^2 \geq A(x)\xi\xi \geq \alpha|\xi|^2, \text{ for a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^d, \tag{1.3}$$

for some $M \geq \alpha > 0$, and f is the source term in $L^1(\Omega)$.

Convection-diffusion problems frequently appear in applied sciences and engineering, being one of the basic problems in mathematical fluid mechanics. Yet, in despite of being a linear problem, its numerical solution faces serious difficulties: linear methods of order two or more fail to satisfy the maximum principle, which however is required to obtain physically admissible solutions. Thus, non-linear techniques are required to obtain high-precision solvers satisfying the maximum principle or, at least, slightly oscillating numerical solutions: Method of characteristics, slope limiters, flux limiters, ENO schemes, etc.

We use in this paper non-linear residual distribution schemes to solve the convection-diffusion equations, in particular the PSI (Positive Streamwise Implicit) method, as an alternative to the more usual methods mentioned above. Residual distribution numerical schemes are both finite volume and piecewise affine finite element schemes. To update the unknown on each grid element, the element residual is distributed among the nodes situated downwind, to achieve stability. Also, the PSI (and some other non-linear residual distribution schemes), are second order accurate. From the point of view of numerical analysis, the PSI method may be cast as a Petrov-Galerkin method. This allows to use the standard tools of functional analysis to perform its stability and error analysis (Cf. [1]).

The situation hardens if one is interested by convection-diffusion equations with data in L^1. This is the case of the equation satisfied by the turbulent kinetic energy in turbulence modeling, and by the heat equation in thermoelectric modeling. As $L^1(\Omega)$ is not imbedded in $H^{-1}(\Omega)$ in two or more space dimensions, the standard analysis theory of problem (1.1) does not hold. Instead this problem admits a unique *renormalized* solution in $W^{1,q}(\Omega)$, $1 \leq q < \frac{d}{d-1}$ (See [3], [4]). We recall the concept of renormalized solution in an Appendix.

In the case of pure diffusion ($a = 0$), the analysis of finite element approximations of problem (1.1) has been recently carried on in [2]. It is proved that the standard Galerkin piecewise affine finite element approximation converges in $W^{1,q}(\Omega)$, $1 \leq q < \frac{d}{d-1}$ to the renormalized solution for suitable triangulations. This is essentially based upon the fact that the discrete diffusion matrix is an M-matrix for such triangulations. Remarkably, this is also a sufficient condition for the discrete maximum principle to be satisfied.

Here we prove that the PSI method provides suitable discretizations of problem (1.1) satisfying the three requirements mentioned: It satisfies the discrete maximum principle, it is of high precision and it provides uniform estimates in $W^{1,q}(\Omega)$, $1 \leq q < \frac{d}{d-1}$. Our analysis is an extension of the mentioned analysis of [2] (for the approximation of pure diffusion equation with low-regularity data), combined with that performed in [1] (for the approximation of the standard convection-diffusion equation by the PSI method). We include error estimates for data in $L^{r,\infty}(\Omega)$ for $1 < r < 2$. Our main innovation is a comparison result in $W^{1,q}(\Omega)$ for approximate solutions of (1.1), which is the main technical contribution of this work.

We also include some numerical relevant tests that seem to indicate that our error estimates are sub-optimal.

2 Discretization

To approximate problem (1.1), let us assume that $\Omega \subset \mathbb{R}^d$ (d =2 or 3) is a polytopic domain. Consider a conforming triangulation $\mathscr{T}_h$ of Ω by triangles if $d = 2$ and tetrahedra if $d = 3$. As usual we assume that h denotes the largest diameter of the elements of

$\mathscr{T}_h$. We consider the finite-dimensional space of piecewise affine finite elements built on $\mathscr{T}_h$:

$$V_h = \{v_h \in \mathscr{C}^0(\overline{\Omega}) / \; v_h|_T \in \mathbb{P}_1 \; \forall T \in \mathscr{T}_h, \; v_h = 0 \quad \text{on } \partial\Omega\}. \tag{2.4}$$

Next, we define the form $a_h : V_h \times V_h \mapsto \mathbb{R}$ as

$$a_h(u_h, v_h) = \int_\Omega (a \cdot \nabla u_h) \Pi_{r_h} v_h + \int_\Omega A \nabla u_h \cdot \nabla v_h. \tag{2.5}$$

Here, Π_{u_h} is an interpolation operator from V_h onto a space of *piecewise constant* functions. It depends on the unknown u_h in a way that we specify below. This dependence makes the method to be non-linear. We shall call it "Distributed Interpolation "operator.

We may now formulate our discrete variational approximation of the convection-diffusion problem (1.1), as follows:

$$\left\{ \begin{array}{l} \text{Find } u_h \in V_h \text{ such that} \\ a_h(u_h, v_h) = \langle f, v_h \rangle \qquad \forall v_h \in V_h. \end{array} \right. \tag{2.6}$$

To define the Distributed Interpolation operator, let us denote by $\{b_j\}_{j=1}^N$ the nodes of the mesh located on $\overline{\Omega} \setminus \partial\Omega$, i.e. the interior nodes. We associate with $\mathscr{T}_h$ and an arbitrary function s_h of V_h a discrete space of piecewise constant functions, denoted by $W_h(s_h)$. This space is defined through its *nodal* basis functions $\lambda_1, \lambda_2, \cdots, \lambda_N$ (also depending on s_h), that we assume known for the time being:

$$W_h(s_h) = \text{ span } \{\lambda_1(s_h), \lambda_2(s_h), \ldots, \lambda_N(s_h)\}. \tag{2.7}$$

The functions λ_j have supports that look for "upwind" information with respect to the velocity field a

The distributed interpolation operator associated to element s_h takes values in W_h, and is defined as follows:

$$\begin{array}{rcl} \Pi_{s_h} : \mathscr{C}^0(\overline{\Omega}) \cap H_0^1(\Omega) & \longrightarrow & W_h \\ z & \longrightarrow & \Pi_{s_h} z = \displaystyle\sum_{i=1}^N z(b_i) \lambda_i. \end{array} \tag{2.8}$$

Π_{s_h} is a bijection from V_h onto W_h.

Each actual residual distribution method is characterized by its associated Distributed Interpolation Operator, through the definition of the basis functions λ_j. To ensure that the constant functions are exactly interpolated, these basis functions must satisfy the following property:

For any element $T \in \mathscr{T}_h$,

$$\lambda_{i_T}|_T \geq 0, \; i = 1, \ldots, d+1, \qquad \sum_{i=1}^{d+1} \lambda_{i_T}|_T = 1, \tag{2.9}$$

where i_T is the global index corresponding to the local index i, $i = 1, \ldots, d+1$ on element T.

3 Analysis of discrete problem

Consider the convection $C(u_h)$ and diffusion D matrices of the discrete problem (2.6), with entries

$$C_{ij}(u_h) = \int_\Omega (a\cdot\nabla\varphi_i)\Pi_{u_h}\varphi_j = \int_\Omega (a\cdot\nabla\varphi_i)\lambda_j(u_h) \tag{3.10}$$

and

$$D_{ij} = \int_\Omega A\nabla\varphi_i\nabla\varphi_j. \tag{3.11}$$

We have the following result for convection matrix, in the particular case of PSI method. The proof can be found in [1]:

Lemma 3.1 *For the PSI method,*

- *$C(s_h)$ is a quasi-M matrix for any $s_h \in V_h$, in the sense that*

$$C_{ii}(s_h)\ge 0, \quad C_{ij}(s_h)\le 0 \quad \textit{if} \quad i\neq j, \quad C_{ii}-\sum_{j\neq i}|C_{ij}|\ge 0, \quad \forall i. \tag{3.12}$$

- *$C(s_h)$ is a continuous function from V_h onto the space of square real matrices of dimension $M\times M$.*

We shall also assume that the diffusion matrix D satisfies:

$$D_{ii}-\sum_{j\neq i}|D_{ij}|\ge 0, \quad \forall i. \tag{3.13}$$

In other words, D is assumed to be a diagonally dominant matrix. This assumption is close to the usual assumption which ensures the discrete maximum principle. In the section 6 of [2] some examples where assumption (3.13) is satisfied are given. This depends on geometrical properties of the grid $\mathscr{T}_h$.

Also, denote by $Q(s_h)$ the convection-diffusion matrix $C(s_h)+D$. Lemma 3.1 and property allow to prove

Lemma 3.2 *Under the hypothesis* (3.13)*, matrix $Q(s_h)$ is an M-matrix and satisfies property (3.13), $\forall s_h \in V_h$*

A first consequence of this Lemma is the coerciveness of form a_h:

$$a_h(v_h,v_h)\ge \alpha\|\nabla v_h\|_2^2, \quad \forall v_h\in V_h.$$

This ensures the existence of solution by Brouwer's Fixed Point Theorem or by a standard compactness argument through linearization.

The solution u_h, however, is not bounded in $H^1(\Omega)$ as the r.h.s. f belongs to $L^1(\Omega)$, which is not embedded in $H^{-1}(\Omega)$. Nevertheless, it is possible to bound u_h in $W^{1,q}$:

Theorem 3.3 *Assume that a and A satisfy* (1.2), (1.3)*, and the hypothesis* (3.13)*. For every $h>0$, let u_h be one solution of problem* (2.6)*, then $\{u_h\}_{h>0}$ is bounded in $W_0^{1,q}$ and there exists a constant $C>0$ independent of h, such that*

$$\|u_h\|_{W_0^{1,q}} \le C\|f\|_{L^1}.$$

This result is a consequence of Theorem 2.1 in [2] and Lemma 3.2.

4 Convergence Analysis and Error Estimates

Convergence of the sequence $\{u_h\}_{h>0}$ to a re-normalized solution of problem (1.1) follows from a comparison result between solutions of the discrete problem when the data are regularized. This result is the main contribution of this work:

Theorem 4.1 *Let u_h be a solution of*

$$\int_\Omega (a\cdot\nabla u_h)\Pi_{u_h} w_h + \int_\Omega A\nabla u_h\cdot\nabla w_h = \int_\Omega f w_h, \quad \forall w_h \in V_h, \tag{4.14}$$

and let u_h^ε be a solution of

$$\int_\Omega (a\cdot\nabla u_h^\varepsilon)\Pi_{u_h^\varepsilon} w_h + \int_\Omega A\nabla u_h^\varepsilon\cdot\nabla w_h = \int_\Omega f^\varepsilon w_h, \qquad \forall w_h \in V_h. \tag{4.15}$$

If the assumptions of Theorem 3.3 hold, $\{u_h^\varepsilon - u_h\}$ is bounded in $W_0^{1,q}$ for every q with $1 \le q < d/(d-1)$,and satisfies

$$\|u_h^\varepsilon - u_h\|_{W_0^{1,q}} \le C\left(\|f^\varepsilon - f\|_1 + h^{2\left(1-\frac{d}{p}\right)}\|a\|_p^2\|f^\varepsilon\|_2^2 + h^2\|f^\varepsilon\|_2^2\right), \tag{4.16}$$

where the constant C is independent of h.

This result allows to prove our main result,

Theorem 4.2 *We retain the assumptions of Theorem 3.3. For every $h>0$, let u_h be a solution of (2.6). Then for every q with $1 \le q < \frac{d}{d-1}$,*

$$\lim_{h\to 0}\|u_h - u\|_{W_0^{1,q}} = 0,$$

where u is the unique renormalized solution of (1.1).

Proof. Let u^ε be the unique solution of problem (1.1) with regularized data $f^\varepsilon \in L^2(\Omega)$. We assume that

$$f^\varepsilon \mapsto f \quad \in L^1(\Omega), \qquad \text{with} \quad \|f^\varepsilon\|_2 \le \frac{1}{\varepsilon}\|f\|_1.$$

The dicretization error can be split as follows:

$$\|u-u_h\|_{W_0^{1,q}} \le \|u-u^\varepsilon\|_{W_0^{1,q}} + \|u^\varepsilon - u_h^\varepsilon\|_{W_0^{1,q}} + \|u_h^\varepsilon - u_h\|_{W_0^{1,q}} = I + II + III.$$

Each term (I), (II), (III) converges to 0 when h and ε go to 0. Indeed, the continuous dependence of the renormalized solution with respect to the data implies that (I) tends to zero (Cf. [F. Murat]). Next, by applying Theorem 7.2 of [1] with data f^ε, we see that $\|u^\varepsilon - u_h^\varepsilon\|_{H_0^1}$ converges to 0, when h goes to 0, for any $\varepsilon > 0$. Therefore (II) also tends to zero.

As $\|f^\varepsilon\|_2 \le \frac{1}{\varepsilon}\|f\|_1$ then (4.16) can be written as:

$$\|u_h^\varepsilon - u_h\|_{W_0^{1,q}} \le C\left(\|f^\varepsilon - f\|_1 + \frac{h^{2\left(1-\frac{d}{p}\right)}}{\varepsilon^2}\|a\|_p^2\|f\|_1^2 + \frac{h^2}{\varepsilon^2}\|f\|_1^2\right), \tag{4.17}$$

By choosing $\varepsilon = \varepsilon_0$ small enough in (4.17), we can make the first term in (4.17) small and next we can adjust $h_0 = h_0(\varepsilon_0)$ such that for all $h \le h_0$ the two remaining terms in (4.17) are also small. Therefore the term (III) converges to 0 with h. ∎

For every r with $1 < r < +\infty$, we denote by $L^{r,\infty}(\Omega)$ the Marcinkiewicz space whose norm is defined by

$$\|v\|_{r,\infty} = \sup_{\lambda>0}(\lambda|\{x\in\Omega : |v(x)| \ge \lambda\}|^{1/r}).$$

Next, error estimates for data $f \in L^{r,\infty}(\Omega)$ may be derived using the techniques introduced in [2]:

Theorem 4.3 *Under the assumptions of Theorem 4.2, and $f \in L^{r,\infty}(\Omega)$ for some r with $1 < r < 2$, we have the error estimate*

$$\|u - u_h\|_{W_0^{1,q}} \le C h^{2\left(1-\frac{1}{r}\right)\left(1-\frac{d}{p}\right)}\|f\|_{r,\infty}. \tag{4.18}$$

5 Numerical tests

We have tested our error estimates in 2D ($d = 2$) for radial data. The domain Ω is the unit ball. We have set

$$f(\mathbf{x}) = \frac{1}{|\mathbf{x}|^{2/r}} \in L^{r,\infty}(\Omega), \quad \text{with} \quad \mathbf{x} = (x_1, x_2), \tag{5.19}$$

$$a(\mathbf{x}) = (x_2, -x_1) \in L^\infty(\Omega)^2. \tag{5.20}$$

We have estimated the convergence order in $W_0^{1,1}(\Omega)$ norm by using three nested non-structured grids with sizes $4h$, $2h$ and h. More precisely,

$$p^h_{W_0^{1,1}(\Omega)} \simeq \frac{\log\left(||u_{4h} - u_{2h}||_{W_0^{1,1}(\Omega)}\right) - \log\left(||u_{2h} - u_h||_{W_0^{1,1}(\Omega)}\right)}{\log(2)},$$

where u_{4h}, u_{2h} and u_h denote the solutions of the discrete problem (2.6) computed on the meshes of sizes $4h$, $2h$ and h, respectively. The next table shows the estimated convergence orders in $W_0^{1,1}(\Omega)$ norm with several values of r and two different grids, corresponding to 395 nodes (intermediate grid, 4h=1/30) and 2 147 nodes (fine grid, 4h=1/46).

r	Theoretical order (4.18)	Intermediate grid	Fine grid
1.1	.182	.138	.142
1.5	.667	.684	.870
1.9	.947	.858	.948

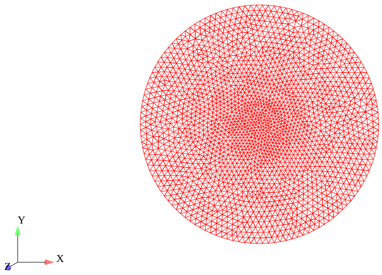

Figure 1: Grid with 2 147 nodes, used to solve discrete problem (2.6).

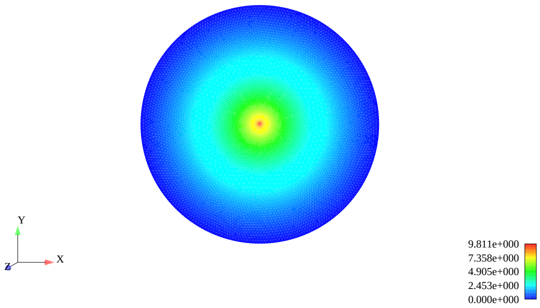

Figure 2: Map of solution of discrete problem (2.6) with data (5.19) with $r = 1.1$ and (5.20).

The finer grid that we have used has some 34 000 nodes. We may observe that the estimates are rather accurate. A mid-size grid with 2 147 nodes is displayed in Figure 1. The density of nodes has been increased near the origin to better solve the singularity. In Figure 2 we display the numerical solution computed on the grid depicted in Figure 1, that indeed is radial, and presents its maximum at the singularity $\mathbf{x} = (0,0)$.

The analysis of the numerical results is more complex when dealing with less smooth velocity fields, and further studies, which are now in progress, are required.

Appendix: Renormalized solution

Let us start by recalling the definition of the renormalized solution for problem (1.1):

Definition 5.1 *A function u is a renormalized solution of* (1.1) *if it satisfies*

- $u \in L^1(\Omega)$
- $\forall k > 0\,, \quad T_k(u) \in H_0^1(\Omega)$, *where* $T_k(u) = \begin{cases} u & |u| \le k, \\ k\frac{u}{|u|} & |u| > k. \end{cases}$
- $\lim_{k\to\infty} \frac{1}{k} \int_{|u|\le k} |\nabla u|^2\, dx = 0$
- $\forall S \in \mathscr{C}_c^1(\mathbb{R}) = \{v \in \mathscr{C}^1(\mathbb{R})$ *with compact support* $\}$, *the equation*

$$(a\cdot\nabla u)S - (\mathrm{div}\,(A\,\nabla\,u))S = fS$$

is satisfied in the distributions sense.

The above definition of renormalized solution was introduced by P.L. Lions & F. Murat [3] (see also [4]). The main interest of the definition of renormalized solution is the following well-posedness theorem (Cf. [4]),

Theorem 5.2 *Assume that a and A respectively satisfy* (1.2) *and* (1.3). *Then there exists a unique renormalized solution of* (1.1). *Moreover,*

$$u \in W_0^{1,q}(\Omega) \textit{ for every } q \textit{ with } 1 \le q < \frac{d}{d-1}.$$

Finally, this unique solution depends continuously on the right-hand side f in the following sense: if f^ε *is a sequence which satisfies* $f^\varepsilon \to f$ *strongly in* $L^1(\Omega)$ *as* $\varepsilon \to 0$, *then the sequence* u^ε *of the renormalized solutions of* (1.1) *for the right-hand sides* f^ε *satisfies for every q with* $1 \le q < \frac{d}{d-1}$,

$$\|u^\varepsilon - u\| \le C\,\|f^\varepsilon - f\|_{L^1} \qquad \text{and so} \qquad \lim_{\varepsilon\to 0} \|u^\varepsilon - u\|_{W_0^{1,q}} = 0.$$

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