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A posteriori error analysis for a large-scale ocean circulation finite element model

Tomás P. Barrios[1], J. Manuel Cascón[2], Galina C. García[3]

[1] *Dpto. de Matemática y Física Aplicada, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile. E-mail:* `tomas@ucsc.cl`.
[2] *Dpto. de Departamento de Matemática, Univ. de Salamanca. E-mail:* `casbar@usal.es`.
[3] *Dpto. de Matemáticas y Ciencia de la Computación, Universidad de Santiago, Casilla 307, Correo 2, Santiago, Chile. E-mail:* `galina@fermat.usach.cl`.

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Resumen

We consider the finite element solution of the stream function and vorticity formulation for a large-scale ocean circulation model. We propose an *a porteriori* estimator to reduce the spurious oscillations and poor resolution which arise when convective terms are dominant. We proof the equivalence, up to higher order terms, between the error and a residual type error estimator. We present several numerical experiments confirming the theoretical properties of the estimator, and illustrating the capability of the corresponding adaptive algorithm to localize the singularities and the large stress regions of the solution.

1. Large scale ocean circulation model

Let us consider the steady-state linear quasi-geostrophic ocean model [3, 9] described by the following equations:

$$\begin{cases} -A_H \Delta \mathbf{u} + \gamma \mathbf{u} + (f_0 + \beta x_2)\, \mathrm{k} \wedge \mathbf{u} + \dfrac{1}{\rho_0} \nabla p = \boldsymbol{\tau} \quad \text{in } \Omega \times (0,T), \\ \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \text{ and } \mathbf{u} = 0 \quad \text{on } \Gamma, \end{cases}$$

where $\mathbf{u} = (u_1, u_2)$ and p are the velocity and the pressure of the fluid at $x = (x_1, x_2) \in \mathbb{R}^2$. In this model, A_H represents the horizontal *eddy viscosity* coefficient, γ is the bottom *friction* coefficient, ρ_0 is the fluid density, $\boldsymbol{\tau}$ is the wind stress, and $(f_0 + \beta x_2)\mathrm{k} \wedge u$ is the Coriolis term, with $\mathrm{k} \wedge \mathbf{u} = (-u_2, u_1)$. We have used the β-plane approximation, with $\beta = 2\Omega_0 R^{-1}\cos\tilde{\theta}_0$, where Ω_0 and R are the angular velocity and radius of the Earth, respectively, and $\tilde{\theta}_0$ a reference latitude. This model describes large-scale horizontal ocean dynamics.

Introducing the stream function and the vorticity, we obtain the following problem (see [4] for details): *Find* $(\omega, \phi) \in H^1(\Omega) \times \Phi$ *satisfying:*

$$\begin{aligned}\epsilon_m(\mathbf{curl}\,\omega, \mathbf{curl}\,\phi)_{[L^2(\Omega)]^2} + \epsilon_s(\omega,\phi)_{L^2(\Omega)} - \left(\frac{\partial\psi}{\partial x_1}, \phi\right)_{L^2(\Omega)} \\ = (\boldsymbol{\tau}, \mathbf{curl}\,\phi)_{[L^2(\Omega)]^2} \qquad \forall\, \phi \in \Phi, \qquad (1)\\ (\omega,\mu)_{L^2(\Omega)} - (\mathbf{curl}\,\psi, \mathbf{curl}\,\mu)_{[L^2(\Omega)]^2} = 0 \qquad \forall\, \mu \in H^1(\Omega), \qquad (2)\end{aligned}$$

where the coefficients ϵ_s and ϵ_m are the non-dimensional Stommel and Munk numbers, respectively:

$$\epsilon_m = \frac{A_H}{\beta L^3}, \quad \epsilon_s = \frac{\gamma}{\beta L}, \qquad (3)$$

with L a representative horizontal length scale of ocean circulation (see [3] and [9] for typical values).

In [4], adapting the arguments of [7] to the presence of a skew-symmetric Coriolis term, we proved the following existence and uniqueness result.

Theorem 1.1 *Let* Ω *be either a convex polygon or such that its boundary* Γ *is of class* $\mathcal{C}^2$*. Then, Problem* (1) *attains a unique solution* $(\omega,\psi) \in H^1(\Omega) \times \Phi$*.*

Let us now describe the finite element scheme. To this end, let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\bar{\Omega}$ by triangles T of diameter h_T and define, as usual, $h := \text{máx}\{\, h_T :\ T \in \mathcal{T}_h \,\}$. Given an integer $l \geq 1$ and a subset S of $\mathbb{R}^2$, we denote by $\mathcal{P}_l(S)$ the space of polynomials in two variables defined on S of total degree at most l, and for each $T \in \mathcal{T}_h$ we define the space of continuous piecewise functions

$$M_h := \{v_h \in C(\overline{\Omega}) : \quad v_h|_T \in \mathcal{P}_l(T) \quad \forall T \in \mathcal{T}_h\},$$

and

$$\Phi_h := M_h \cap \Phi\,.$$

The corresponding Galerkin scheme is: *Find* $(\omega_h, \phi_h) \in M_h \times \Phi_h$ *such that:*

$$\begin{aligned}\epsilon_m(\mathbf{curl}\,\omega_h, \mathbf{curl}\,\phi_h)_{[L^2(\Omega)]^2} + \epsilon_s(\omega_h,\phi_h)_{L^2(\Omega)} - \left(\frac{\partial\psi_h}{\partial x_1}, \phi_h\right)_{L^2(\Omega)} \\ = (\boldsymbol{\tau}, \mathbf{curl}\,\phi_h)_{[L^2(\Omega)]^2} \quad \forall\, \phi_h \in \Phi_h, \\ (\omega_h,\mu_h)_{L^2(\Omega)} - (\mathbf{curl}\,\psi_h, \mathbf{curl}\,\mu_h)_{[L^2(\Omega)]^2} = 0 \quad \forall\, \mu_h \in M_h.\end{aligned} \qquad (4)$$

Here, we just mention that the proof of the well posedness of (4) as well as the corresponding *a priori* error estimates, follows the arguments developed in [7] for the Stokes problem. For the sake of completeness, we included the next theorem, which establishes the *a priori* error estimates for the mixed scheme.

Theorem 1.2 *Assume that the solution of continuous problem* $(\omega, \psi) \in H^s(\Omega) \times H^{s+1}(\Omega)$, *for* $1 \leq s \leq l$. *Then the discrete scheme* (4) *attains a unique solution* $(\omega_h, \psi_h) \in M_h \times \Phi_h$, *and if* $l > 1$, *there holds*

$$\|\omega - \omega_h\|_{L^2(\Omega)} + |\psi - \psi_h|_{H^1(\Omega)} \leq ch^{s-1}(\|\omega\|_{H^s(\Omega)} + \|\psi\|_{H^{s+1}(\Omega)})$$

In addition, when $l = 1$ *and* $\psi \in H^3(\Omega)$, *then, for all* $\epsilon > 0$

$$|\psi - \psi_h|_{H^1(\Omega)} \leq C(\epsilon) h^{1-\epsilon} \|\psi\|_{H^3(\Omega)} ,$$

and

$$\|\omega - \omega_h\|_{L^2(\Omega)} \leq C(\epsilon) h^{1/2-\epsilon} \|\psi\|_{H^3(\Omega)} .$$

Proof: See Theorem 4 in [4].

2. A posteriori error analysis

In this section, we use a duality technique and derive a reliable and efficient residual-based *a posteriori* error estimate for the stream function and vorticity formulation of the Stokes problem. Let us first introduce some notations. Given $T \in \mathcal{T}_h$, we denote by $E(T)$ the set of its edges, and by E_h the set of all edges of the triangulation $\mathcal{T}_h$. Then we can write $E_h = E_h(\Omega) \cup E_h(\Gamma)$, where $E_h(\Omega) := \{e \in E_h : \ e \subseteq \Omega\}$, $E_h(\Gamma) := \{e \in E_h : \ e \subseteq \Gamma\}$. In what follows, h_T and h_e stand for the diameters of triangle $T \in \mathcal{T}_h$ and edge $e \in E_h$, respectively. Also, given a vector valued function $\mathbf{v} := (v_1, v_2)^{\mathbf{T}}$ defined in Ω, an edge $e \in E(T) \cap E_h(\Omega)$, and the unit outward normal vector $\boldsymbol{\nu}_T$ along e, we let $J[\mathbf{v} \cdot \boldsymbol{\nu}_T]$ be the corresponding jump across e, that is $J[\mathbf{v} \cdot \boldsymbol{\nu}_T] := (\mathbf{v}_T - \mathbf{v}_{T'})|_e \cdot \boldsymbol{\nu}_T$, where T' is the other triangle of $\mathcal{T}_h$ having e as edge.

The following theorem is the main result of the present section.

Theorem 2.1 *Let* $(\omega, \psi) \in H^1(\Omega) \times \Phi$ *and* $(\omega_h, \psi_h) \in M_h \times \Phi_h$ *be the unique solution of continuous and discrete formulation* (1) *and* (4), *respectively. Assume that the data* $\boldsymbol{\tau} \in [H^1(\Omega)]^2$. *Then there exist* $C_{\mathtt{eff}}$, $C_{\mathtt{rel}} > 0$, *independent of* h, *such that*

$$\|(\omega - \omega_h, \psi - \psi_h)\|^2_{L^2(\Omega) \times H^1_0(\Omega)} \leq C_{\mathtt{rel}} \sum_{T \in \mathcal{T}_h} \eta_T^2 ,$$

and

$$\begin{aligned} C_{\mathtt{eff}} \eta_T^2 \ \ &\leq \|(\omega - \omega_h, \psi - \psi_h)\|^2_{L^2(\Delta(T)) \times H^1_0(\Delta(T))} + h_T^2 \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{[L^2(T)]^2} \\ &\quad + \ \ h_T^4 \| \operatorname{curl}(\boldsymbol{\tau} - \boldsymbol{\tau}_h)\|_{L^2(T)}, \end{aligned}$$

where for any $T \in \mathcal{T}_h$ *we define*

$$\eta_T^2 \quad := h_T^4 \, \|\epsilon_m \Delta \omega_h - \epsilon_s \omega_h + \tfrac{\partial \psi_h}{\partial x_1} + \operatorname{curl} \boldsymbol{\tau}\|^2_{L^2(T)} \ + \ h_T^2 \, \|\omega_h + \Delta \psi_h\|^2_{L^2(T)} \tag{5}$$

$$+ \textstyle\sum_{e \in E(T) \cap E_h} h_e \|J[\nabla(\psi_h) \cdot \boldsymbol{\nu}_T]\|^2_{L^2(e)} + \sum_{e \in E(T) \cap E_h(\Omega)} h_e^3 \|J[\nabla(\omega_h) \cdot \boldsymbol{\nu}_T]\|^2_{L^2(e)} . \tag{6}$$

and $\boldsymbol{\tau}_h$ *denotes a suitable approximation of* $\boldsymbol{\tau}$.

In order to proof Theorem 2.1, we extended the arguments of [2] to the quasi-geostrophic oecan model. Our approach is based on the introduction of a dual problem technique, we deduce practically the same error indicator developed in [1] for the Stokes system. It only differs in appropriate mesh-size factors, which, under appropriate assumptions, allow us to prove efficiency in natural norms. In this sense, it can be seen as an extension of the applicability of the error indicator developed in [1] to the standard stream function and vorticity formulation.

3. Numerical Experiments

In this section we present some numerical examples which confirm the reliability and efficiency of our *a posteriori* estimator, which exhibits an optimal convergence rate. The experiments have been performed with the finite element toolbox ALBERTA using refinement by recursive bisection [10], and the solution of the corresponding discrete system has been computed using SuperLU library [5].

We propose the following standard adaptive finite element method (AFEM)

$$\mathsf{SOLVE} \rightarrow \mathsf{ESTIMATE} \rightarrow \mathsf{MARK} \rightarrow \mathsf{REFINE}.$$

Hereafter, we replace the subscript h by k, where k is the counter of the adaptive loop, and define $h_k := \text{máx}\{h_T : T \in \mathcal{T}_k\}$. Then, given a mesh $\mathcal{T}_k$, the procedure SOLVE is an efficient direct solver for computing the discrete solution (ω_k, ψ_k), ESTIMATE calculates the error indicators, $\eta_k(T)$ for all $T \in \mathcal{T}_k$ depending on the computed solution and data. Based on the values of $\{\eta_k(T)\}_{T\in\mathcal{T}_k}$, the MARK procedure generates a set of marked elements subject to refinement. For the elements selection, we rely on Dörfler marking [6]: Fixed a marking parameter $\theta \in (0, 1]$, the procedure MARK outputs a subset of *marked elements* $\mathcal{M}_k \subset \mathcal{T}_k$, *ie.* such that $\mathcal{M}_k$ satisfies Dörfler's property

$$\eta_k^2(\mathcal{M}_k) \geq \theta^2\, \eta_k^2(\mathcal{T}_k). \tag{7}$$

The idea is to select subsets of the triangulation $\mathcal{T}_k$ whose element contributions sum up to a fixed amount of the total. In practice, the subsets are chosen as small as possible by collecting the biggest values in order to only introduce a small number of new degrees of freedom (DOF) in the next mesh.

Finally, the procedure REFINE creates a conforming refinement $\mathcal{T}_{k+1}$ of $\mathcal{T}_k$, bisecting b times at least, all the elements of marked set $\mathcal{M}_k$.

For our experiments we have taken the usual values for θ and b, that is $\theta = 0{,}6$ and $b = 2$ (see [10, 8]) and we use non-structured meshes.

In the first experiment, we will compare the performance of a finite element method based on uniform refinement (FEM) with the adaptive method (AFEM) that we have described before for the Stokes problem. As commented in Section 1, the *a priori* error estimate for the stream function and vorticity formulation on quasi-uniform meshes, is of order $\mathcal{O}(h^{l-1})$ (where l denotes the degree) for $l > 1$, and of order $\mathcal{O}(h^{1/2})$ if $l = 1$ and ψ has some extra regularity, *i.e.*

$$\text{Total Error} := |\psi - \psi_k|_{H^1(\Omega)} + \|\omega - \omega_k\|_{L^2(\Omega)} \leq \begin{cases} C(\psi,\omega)h_k^{l-1}, & l > 1 \\ C(\psi,\omega)h_k^{1/2} & l = 1. \end{cases} \tag{8}$$

Note that, this result is only for uniform refinement. In order to compare uniform and adaptive meshes, we introduce the *empirical order of convergence* for the Total Error in step k as,

$$\mathrm{EOC}_k = 2\frac{\log\left(\text{Total Error}_{k-1}/\text{Total Error}_k\right)}{\log(\mathrm{DOFs}_k/\mathrm{DOFs}_{k-1})},$$

where DOFs_k denotes the number of degrees of freedom in mesh $\mathcal{T}_k$. Respectively, $\mathrm{EOC}_k(\psi)$ and $\mathrm{EOC}_k(\omega)$ can also be defined, with their corresponding norms. It is easy to check, that if the order of the method on quasi-uniform meshes is $\mathcal{O}(h^{l-1})$, then $\mathrm{EOC}_k = l-1$. We also analyze the decay of the Total Error (or estimator) in terms of DOFs. Note that in quasi-uniform meshes and two dimensions, we have $\#DOFs^{-1/2} \approx h$.

3.1. Example 1: Stokes Problem

We consider the following Stokes problem with a known solution on a convex domain (for the *a posteriori* analysis see [2]):

$$\nu(\mathbf{curl}\,\omega, \mathbf{curl}\,\phi)_{[L^2(\Omega)]^2} = (\mathbf{f}, \mathbf{curl}\,\phi)_{[L^2(\Omega)]^2} \qquad \forall\,\phi \in \Phi, \tag{9}$$

$$(\omega, \mu)_{L^2(\Omega)} - (\mathbf{curl}\,\psi, \mathbf{curl}\,\mu)_{[L^2(\Omega)]^2} = 0 \qquad \forall\,\mu \in H^1(\Omega). \tag{10}$$

Let $\Omega = [0,1]^2$, $\nu = 1$, and a right hand side such that the exact solution (ψ, ω) is:

$$\psi = \left((1-x)\,(1-e^{-20x})\sin \pi y\right)^2, \qquad \omega = -\Delta\psi.$$

In Fig. 1, we show the initial mesh (left) and the meshes obtained by the AFEM in two different iterations for piecewise linear elements. It is clear that our method localizes the boundary layer and concentrates the refinement in this region.

In Fig. 2 we report a comparison between uniform and adaptive refinement. In this example, the improvement of the AFEM for P2 is more pronunciate. For degreel l bigger than one, while the EOC of FEM looks stabilized at $l-1$, the AFEM presents optimal order: Total Error $\sim\mathrm{DOFs}^{-l/2}$. In the case of piecewise linear element, the AFEM has a better behaviour in the first steps, because the boundary layer is quickly localized, however asymptotically the result is exactly the same.

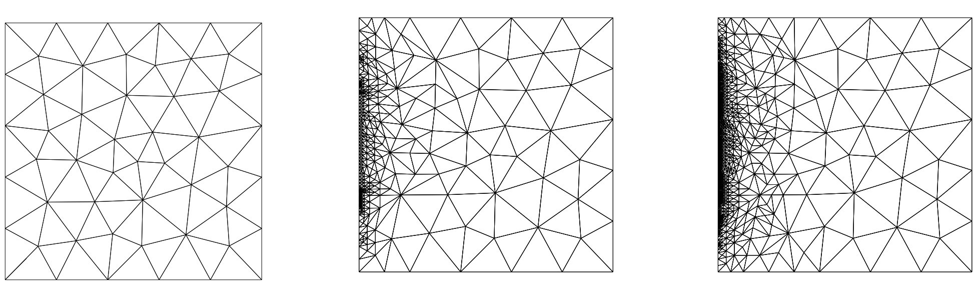

Figura 1: Example 2. Meshes generated by AFEM with piecewise linear elements: *left*-Initial mesh, *center*-20th iteration (455 DOFs), *right*-60th iteration (4343 DOFs).

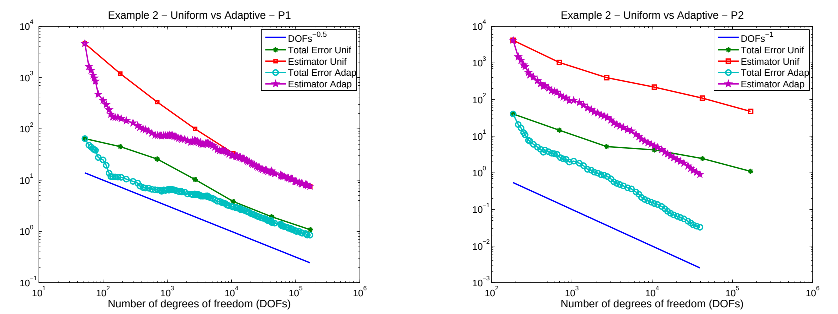

Figura 2: Example 2. Decay of the total error and estimator vs DOFs for uniform and adaptive refinement, and elements of degrees 1 and 2.

3.2. Quasi-geostrophic ocean model

The goal of our experiment is to apply the AFEM to a large-scale ocean model where a typical phenomenon is the formation of the Western boundary currents. Because of this, adaptive strategies are useful for the numerical solution of the quasi-geostrophic model.

In this example, we consider the solution of (4), with realistic values for the Munk and Stommel numbers (see [9]):

$$\epsilon_m = 6 \times 10^{-5} \qquad \epsilon_s = 0{,}05,$$

which correspond to

$$\gamma = 7 \times 10^{-6} s^{-1}, \qquad A_H = 1{,}2 \times 10^3 m^2 s^{-1}, \qquad L = 10^6 m, \qquad D_0 = 800m.$$

For the wind stress, we use (see also [9]):

$$\boldsymbol{\tau} = \left(-\frac{1}{\pi} \cos \pi x, 0 \right).$$

We consider two different domains. The fist one is the unit square, and the second one is L-shaped domains. Figs. 4 and 5 show an intermediate mesh and the corresponding stream lines solution obtained for piecewise linear elements, for convex and non convex respectively.

In this case there is no analytical solution available. However, Fig. 3 shows that the estimated errors attain an optimal convergence rate. Let us remark that although our theory does not cover the non convex domain, the proposed AFEM behaves as well as for the convex one.

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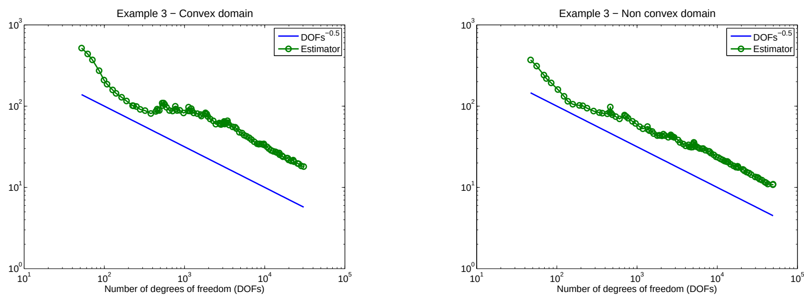

Figura 3: Decay of the estimator vs DOFs for piecewise linear functions.

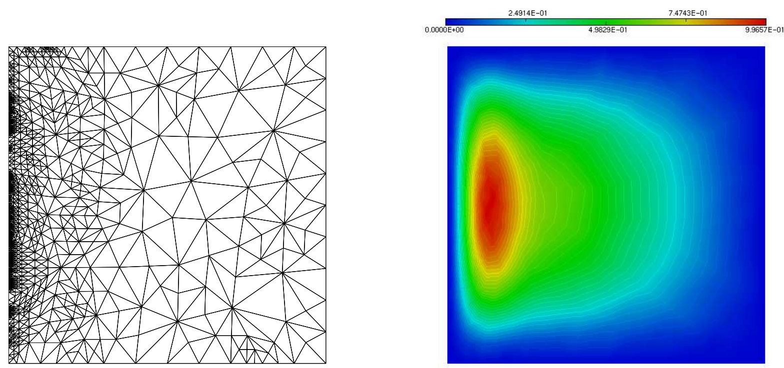

Figura 4: Mesh and stream function obtained by AFEM with piecewise linear elements in the 50th iteration (2454 DOFs)

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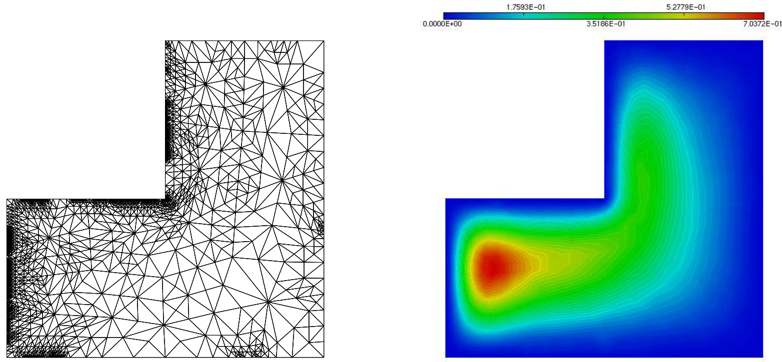

Figura 5: L-shape. Mesh and stream function obtained by AFEM with piecewise linear elements in 50th iteration (4031 DOFs)

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