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A locking free FEM in optimal control of Timoshenko beam

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Resumen

In this work we analyze the numerical approximation of an optimal control problem of a Timoshenko beam. In order to avoid locking, we focus on the finite element method used to compute the beam behavior, to minimize it. Optimal order error estimates are obtained for the control variable, which is the amplitude of secondary forces modeled as Dirac's delta distributions. These estimates are valid with constants that do not depend on the thickness of the beam. Moreover, an interpolation postprocessing technique is proposed to the approximations and it is proved that these approximations have superconvergence order. In order to assess the performance of the method, numerical tests are reported.

1. State Equation

In this section we introduce the *state equations* in the control problems considered.

Let us consider an elastic beam of thickness $t \in (0,1]$, with reference configuration $\mathbf{I} \times (-t/2, t/2)$, where $\mathbf{I} := (0, L)$ with L the length of the beam. The deformation of the beam in the frequency domain is described by means of the Timoshenko model in terms of the rotations amplitude θ of its midplane and the transverse displacement amplitude w (see [5]). Assuming that the beam is clamped, considering $f, g \in H^{-1}(\mathbf{I})$, and $v, \beta \in H_0^1(\mathbf{I})$ as test functions, the beam deformation is solution of the following problem:

Find $(w_t, \theta_t) \in H_0^1(\mathbf{I})^2$ *such that*

$$\left\{ \begin{array}{l} \dfrac{E}{12} \displaystyle\int_{\mathbf{I}} \frac{d\theta_t}{dx} \frac{d\beta}{dx} dx + \frac{\kappa}{t^2} \int_{\mathbf{I}} \left(\frac{dw_t}{dx} - \theta_t \right) \left(\frac{dv}{dx} - \beta \right) dx - \omega_t^2 \int_{\mathbf{I}} w_t v dx \\ -\omega_t^2 \dfrac{t^2}{12} \displaystyle\int_{\mathbf{I}} \theta_t \beta dx = \langle f, v \rangle + \frac{t^2}{12} \langle g, \beta \rangle, \qquad \forall (v, \beta) \in H_0^1(\mathbf{I})^2, \end{array} \right. \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the dual parity between the spaces $H^{-1}(\mathbf{I})$ and $H_0^1(\mathbf{I})$, $\omega_t^2 := \rho\widehat{\omega}^2/t^2$ is the rescaled angular frequency, and $\kappa = Ek/2(1+\bar{\nu})$.

By $a_{\omega t}$, we denote the bilinear continuous form in $H_0^1(\mathbf{I})^2$ that appear in the left hand side of (1):

$$a_{\omega t}\left((w_t,\theta_t),(v,\beta)\right) = \frac{E}{12}\int_{\mathbf{I}}\frac{d\theta_t}{dx}\frac{d\beta}{dx}dx + \frac{\kappa}{t^2}\int_{\mathbf{I}}\left(\frac{dw_t}{dx}-\theta_t\right)\left(\frac{dv}{dx}-\beta\right)dx$$
$$-\omega_t^2\int_{\mathbf{I}} w_t v dx - \omega_t^2\frac{t^2}{12}\int_{\mathbf{I}}\theta_t\beta dx,\ \forall (w_t,\theta_t),(v,\beta)\in H_0^1(\mathbf{I})^2. \tag{2}$$

The bilinear form $a_{\omega t}$ is not positive definite, for this reason *Lax-Milgram Lemma* can not be applied to obtain existence and uniqueness of solution of the variational problem (1). However, it is clear that problem (1) has a unique solution if ω_t^2 is not a eigenvalue of the homogeneous problem:

Find $(w_t,\theta_t)\in H_0^1(\mathbf{I})^2$ *such that*

$$a_{\omega t}\left((w_t,\theta_t),(v,\beta)\right) = 0, \quad \forall (v,\beta)\in H_0^1(\mathbf{I})^2. \tag{3}$$

This eigenvalue problem has been recently analyzed in [5], as a particular case of a more general problem, it is proved that the spectrum consists of a sequence of finite multiplicity real eigenvalues converging to infinite.

Let $\omega_t\in\mathbb{R}$ such that $\omega_t^2\notin\mathcal{S}$, where $\mathcal{S}$ denote the spectrum of the problem (3), $t\in(0,1]$ and $f,g\in H^{-1}(\mathbf{I})$. Then, the problem (1) has existence and uniqueness of solution $(w_t,\theta_t)\in H_0^1(\mathbf{I})^2$, moreover the following estimate holds:

$$\|(w_t,\theta_t)\|_{H^1(\mathbf{I})^2} \le C_t\left(\|f\|_{H^{-1}(\mathbf{I})} + t^2\|g\|_{H^{-1}(\mathbf{I})}\right). \tag{4}$$

To avoid the numerical-locking in the static case, we will use a mixed formulation (see [1, 5]) to obtain a locking-free scheme that applies to our dynamic problem. In order to achieve this purpose it is necessary to obtain a stability condition with a constant than does not degenerate when the thickness t goes to zero. For this reason, we introduce the following load problem associated to the Timoshenko equations in static case, written in the mixed form:

Find $(w^t,\theta^t,\gamma^t)\in H_0^1(\mathbf{I})^2\times L^2(\mathbf{I})$ *such that*

$$\left\{\begin{array}{rcll}\dfrac{E}{12}\displaystyle\int_{\mathbf{I}}\frac{d\theta^t}{dx}\frac{d\beta}{dx}dx + \int_{\mathbf{I}}\gamma^t\left(\frac{dv}{dx}-\beta\right)dx & = & \langle f,v\rangle + \dfrac{t^2}{12}\langle g,\beta\rangle, & \forall (v,\beta)\in H_0^1(\mathbf{I})^2,\\ \dfrac{t^2}{\kappa}\displaystyle\int_{\mathbf{I}}\gamma^t\eta dx & = & \displaystyle\int_{\mathbf{I}}\left(\frac{dw^t}{dx}-\theta^t\right)\eta dx, & \forall \eta\in L^2(\mathbf{I}).\end{array}\right.$$

According to [5], as a particular case, this problem has a unique solution $(w^t,\theta^t,\gamma^t)\in H_0^1(\mathbf{I})^2\times L^2(\mathbf{I})$ and there holds:

$$\|w^t\|_1 + \|\theta^t\|_1 + \|\gamma^t\|_0 \le C\left(\|f\|_{H^{-1}(\mathbf{I})} + t^2\|g\|_{H^{-1}(\mathbf{I})}\right). \tag{5}$$

For the numerical approximation, following [1], we consider a family $\{\mathcal{T}_h\}$ of partitions of the interval $\mathbf{I}$:

$$\mathcal{T}_h : 0 = s_0 < s_1 < \cdots s_n = L,$$

with mesh-size

$$h := \max_{j=1,\dots,n} (s_j - s_{j-1}) .$$

We define the following finite element spaces:

$$\mathcal{V}_h := \left\{ v \in H_0^1(\mathbf{I}) : v|_{[s_{j-1},s_j]} \in \mathbb{P}_1, j = 1, \dots, n \right\} \subset H_0^1(\mathbf{I})$$

and

$$\mathcal{W}_h := \left\{ \frac{dv}{dx} + c : \ v \in \mathcal{V}_h, c \in \mathbb{R} \right\} \subset L^2(\mathbf{I}).$$

Thus we can write the discrete version of variational problem (1) as follows:

Find $(w_{th}, \theta_{th}) \in \mathcal{V}_h^2$ *such that*

$$a_{\omega th} \left((w_{th}, \theta_{th}), (v_h, \beta_h) \right) = \langle f, v_h \rangle + \frac{t^2}{12} \langle g, \beta_h \rangle \qquad \forall \ (v_h, \beta_h) \in \mathcal{V}_h^2, \tag{6}$$

where the bilinear form $a_{h\omega t}$ is given by:

$$\begin{aligned} a_{\omega th} \left((w_{th}, \theta_{th}), (v_h, \beta_h) \right) = {} & \frac{E}{12} \int_{\mathbf{I}} \frac{d\theta_{th}}{dx} \frac{d\beta_h}{dx} dx + \frac{\kappa}{t^2} \int_{\mathbf{I}} \pi_h \left(\frac{dw_{th}}{dx} - \theta_{th} \right) \pi_h \left(\frac{dv_h}{dx} - \beta \right) dx \\ & - \omega_t^2 \int_{\mathbf{I}} w_{th} v_h dx - \omega_t^2 \frac{t^2}{12} \int_{\mathbf{I}} \theta_{th} \beta_h dx, \ \forall (w_{th}, \theta_{th}), (v_h, \beta_h) \in \mathcal{V}_h^2, \end{aligned} \tag{7}$$

where π_h denotes the L^2-projector onto $\mathcal{W}_h$.

As in the continuous case, (w_h^t, θ_h^t) denotes the solution of the discrete version of the mixed problem, i.e.:
Find $(w_h^t, \theta_h^t, \gamma_h^t) \in \mathcal{V}_h^2 \times \mathcal{W}_h$ *such that*

$$\left\{ \begin{array}{rcl} \dfrac{E}{12} \displaystyle\int_{\mathbf{I}} \frac{d\theta_h^t}{dx} \frac{d\beta_h}{dx} dx + \int_{\mathbf{I}} \gamma_h^t \left(\frac{dv_h}{dx} - \beta_h \right) dx & = & \langle f, v_h \rangle + \dfrac{t^2}{12} \langle g, \beta_h \rangle , \\ \dfrac{t^2}{\kappa} \displaystyle\int_{\mathbf{I}} \gamma_h^t \eta_h dx & = & \displaystyle\int_{\mathbf{I}} \left(\frac{dw_h^t}{dx} - \theta_h^t \right) \eta_h dx, \end{array} \right.$$

$\forall \ (v_h, \beta_h) \in \mathcal{V}_h^2$ and $\forall \ \eta_h \in \mathcal{W}_h$, respectively.

The following result as been proved in [3]:
THEOREM 1: Given $f, g \in H^{-1}(\mathbf{I})$ and $\omega_t^2 \in \mathbb{R}$ such that $\omega_t^2 \notin \mathcal{S}$. Then, there exists $h_0 > 0$ such that, for all $h < h_0$, the problem (6) has a unique solution (w_{th}, θ_{th}). Moreover, if $f, g \in L^2(\mathbf{I})$, the following estimates hold:

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{H^1(\mathbf{I})^2} \leq Ch \left(\|f\|_{L^2(\mathbf{I})} + t^2 \|g\|_{L^2(\mathbf{I})} \right), \tag{8}$$

$$\|(w_,\theta_t) - (w_{th}, \theta_{th})\|_{L^2(\mathbf{I})^2} \leq Ch^2 \left(\|f\|_{L^2(\mathbf{I})} + t^2 \|g\|_{L^2(\mathbf{I})} \right). \tag{9}$$

On the other hand, if $f = \delta_x$ and $g = \delta_y$, where $x, y \in \mathbf{I}$ are grid-point, then

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{H^1(\mathbf{I})^2} \leq Ch, \tag{10}$$

$$\|(w_t, \theta_t) - (w_{th}, \theta_{th})\|_{L^2(\mathbf{I})^2} \leq Ch^2. \tag{11}$$

2. Control Problem

We introduce the control problems that concern us, for the mathematical optimal control framework we use the notation and the context introduced in [**?**]. For the sake of simplicity, we describe the two kind of sensors, punctual and distributed, jointly.

PUNCTUAL(OR DISTRIBUTED) SENSORS: The problem of active vibration control consists in reducing the vibration in M given points called *punctual sensors*(or along the beam, or on a segment of it, namely (a_1, a_2), with $0 \leq a_1 < a_2 \leq L$, called *distributed sensors*). In order to state the problem mathematically, we make the following choices:

(A.1) the *state of the system* is given by the transversal displacement $w(x)$ of the beam;

(A.2) the *control variable* $\boldsymbol{u}$ is the vector of reals amplitudes of actuators,

$$\boldsymbol{u} = (u_1, \ldots, u_N) \in \mathbb{R}^N,$$

which define the source term f in the problem (1).

(A.3) the set of *admissible controls* is a convex, not empty, closed set $\mathrm{U}_{ad} \subset \mathbb{R}^N$;

(A.4) the *model of the system* that relates the control variable with the state is the Timoshenko problem, i.e. problem (1);

(A.5) (Punctual Case) the observation $\boldsymbol{z}$ is the set of displacement values at M sensors located at given points $p_1, \ldots, p_M \in \mathbf{I}$,

$$\boldsymbol{z}(\boldsymbol{u}) := (w(\boldsymbol{u}, p_1), \ldots, w(\boldsymbol{u}, p_M)) \in \mathbb{R}^M,$$

where, for $\boldsymbol{u} \in \mathbb{R}^N$, $w(\boldsymbol{u}, \cdot)$ denotes the solution of problem (1);
(Distributed Case) the observation $\boldsymbol{y}$ is the transverse displacement in (a_1, a_2), i.e, $\boldsymbol{y}(\boldsymbol{u}, x) = w(\boldsymbol{u}, x)|_{(a_1,a_2)}$.

(A.6) (Punctual Case) the *cost function* to be minimized depends on the observation and eventually on the cost of the control itself, namely,

$$J(\boldsymbol{u}) \quad := \quad \frac{1}{2} \|\boldsymbol{z}(\boldsymbol{u}) - \boldsymbol{z}_d\|_2^2 + \frac{\nu}{2} \|\boldsymbol{u}\|_2^2, \tag{12}$$

where, $\nu \geq 0$ denotes a weighting factor that represents the cost of the control, $\|\cdot\|_2$ is the Euclidian norm in $\mathbb{R}^N$ or $\mathbb{R}^M$ and $\boldsymbol{z}_d$ denotes the desired state, which in our case will be considerer $\boldsymbol{z}_d = \mathbf{0}$.
(Distributed Case) the cost function to be minimized in this problem is

$$\mathcal{J}(\boldsymbol{u}) \quad := \quad \frac{1}{2} \|\boldsymbol{y}(\boldsymbol{u}) - \boldsymbol{y}_d\|_{L_2(a_1,a_2)}^2 + \frac{\nu}{2} \|\boldsymbol{u}\|_2^2, \tag{13}$$

where, $\boldsymbol{y}_d$ denotes the desired state, which in our case is $\boldsymbol{y}_d = \mathbf{0}$.

Thus, the *optimal control problem* will be:

Find $\boldsymbol{u}^{op} \in \mathrm{U}_{ad}$ *such that*

$$J(\boldsymbol{u}^{op}) = \inf_{\boldsymbol{u}\in \mathrm{U}_{ad}} J(\boldsymbol{u}). \tag{14}$$

In both optimal control problems, any solution $\boldsymbol{u}^{op}$ of the minimization problem will be called an *optimal control*. This depends directly on the amplitude of the transverse displacement $w(x)$ of a Timoshenko beam, for this reason, the mathematical analysis of the problem (1) will be considered in what follows.

Both optimal control problems can be posed simultaneously, by considering the following, more general, cost functional:

$$\mathfrak{J}(\boldsymbol{u}) \;:=\; \frac{1}{2}\left\|\boldsymbol{p}(\boldsymbol{u}) - \boldsymbol{p}_d\right\|_{\mathcal{H}}^2 + \frac{\nu}{2}\left\|\boldsymbol{u}\right\|_2^2, \tag{15}$$

where, $\mathcal{H} = \mathbb{R}^M$, $\boldsymbol{p}(\boldsymbol{u}) = \boldsymbol{z}(\boldsymbol{u})$ in the case of the Punctual Sensor and $\mathcal{H} = L^2(a_1, a_2)$, $\boldsymbol{p}(\boldsymbol{u}) = \boldsymbol{y}(\boldsymbol{u})$ in the case of Distributed Sensor. Thus, considering $\boldsymbol{p}_d = \boldsymbol{0}$ the optimal control problems is:

Find $\boldsymbol{u}^{op} \in \mathrm{U}_{ad}$ *such that*

$$\mathfrak{J}(\boldsymbol{u}^{op}) = \inf_{\boldsymbol{u}\in \mathrm{U}_{ad}} \mathfrak{J}(\boldsymbol{u}). \tag{16}$$

For the discrete approximation, let w_{th} be the transversal displacement solution of the discrete problem (6), with a load source f.

Let us introduce, for $i = 0, \ldots, N$, the approximate observation $\boldsymbol{p}_{hi}$ which is obtained by replacing w_{ti} by w_{thi}. Then, the global observation $\boldsymbol{p}_h$ in problem (16) can be written in the following way:

$$\boldsymbol{p}_h(\boldsymbol{u}) := \boldsymbol{p}_{h0} + \sum_{i=1}^{N} u_i \boldsymbol{p}_{hi}.$$

Now, let $\boldsymbol{P}_h \in \mathbb{R}^{N\times N}$ and $\boldsymbol{b}_h$ be defined by

$$\begin{aligned}
(\boldsymbol{P}_h)_{ij} &:= (\boldsymbol{p}_{hj}, \boldsymbol{p}_{hi})_{\mathcal{H}}, \qquad i, j = 1, \ldots, N,\\
(\boldsymbol{b}_h)_i &:= (\boldsymbol{p}_{h0}, \boldsymbol{p}_{hi})_{\mathcal{H}}, \qquad i = 1, \ldots, N.
\end{aligned}$$

The approximate cost function can be written as:

$$\begin{aligned}
\mathfrak{J}_h(\boldsymbol{u}) &= \frac{1}{2}\|\boldsymbol{p}_h(\boldsymbol{u})\|^2 + \frac{\nu}{2}\|\boldsymbol{u}\|^2\\
&= \frac{1}{2}\left\{ \left(\left(\boldsymbol{P}_h + \nu \mathbf{I}\right)\boldsymbol{u}, \boldsymbol{u}\right) + 2\left(\boldsymbol{u}, \boldsymbol{b}_h\right) + \|\boldsymbol{p}_{h0}\|_2^2 \right\}.
\end{aligned} \tag{17}$$

These definitions lead us to the following discrete optimal control problem:

Find $\boldsymbol{u}_h^{op}$ *such that*

$$\mathfrak{J}_h(\boldsymbol{u}_h^{op}) = \inf_{\boldsymbol{u}\in \mathrm{U}_{ad}} \frac{1}{2}\left\{ \left(\left(\boldsymbol{P}_h + \nu \mathbf{I}\right)\boldsymbol{u}, \boldsymbol{u}\right) + 2\left(\boldsymbol{u}, \boldsymbol{b}_h\right) + \|\boldsymbol{p}_{h0}\|_2^2 \right\}. \tag{18}$$

Then, as shown in [3], the following theorem holds:

THEOREM 2

Let us assume that $\nu > 0$ or $\nu \geq 0$ and $\boldsymbol{p}(\boldsymbol{u})$ is one-to-one. If $\mathbf{0} \in \mathrm{U}_{ad}$, then there exists a positive constant C independent of t and h_0 such that, for all $h \in (0, h_0]$ holds,

$$\|\boldsymbol{u}^{op} - \boldsymbol{u}_h^{op}\| \leq Ch.$$

for the *Punctual Sensor* problem.

$$\|\boldsymbol{u}^{op} - \boldsymbol{u}_h^{op}\| \leq Ch^2.$$

for the *Distributed Sensor* problem.

3. Numerical Examples

We consider representative examples of each control scheme. In order to quantify the effect of the control we use the following attenuation measure in both cases:

$$\text{Attenuation (dB)} = -10 \log \left(\frac{\mathbf{J}(\boldsymbol{u})}{\mathbf{J}(\mathbf{0})} \right),$$

where $\mathbf{J}$ denotes the corresponding functional of each problem.

Punctual Sensors: In the Table 1, we report the optimal control computed for different thickness t and successively refined meshes. It also includes the computed order of convergence and the corresponding extrapolated optimal control, obtained by means of a least squares fitting of the model

$$\boldsymbol{u}_h^{op} \approx \boldsymbol{u}_{ex} + Ch^t.$$

Moreover in this table it can be seen that the order of convergence remains uniformly optimal with respect to t, this confirms that the method is locking-free.

Tabla 1: Optimal controls for a punctual sensor problem(scaled by a factor 10^{11}).

	$h = 1/80$	$h = 1/100$	$h = 1/120$	ext	order
$t = 0{,}8$	1.6288181	1.6252264	1.6228266	1.6106965	0.99
$t = 0{,}1$	2.1995987	2.2139872	2.2236505	2.2741965	0.96
$t = 0{,}05$	2.2237015	2.2389812	2.2492472	2.3029311	0.96
$t = 0{,}01$	2.2305049	2.2460195	2.2564446	2.3109559	0.96
$t = 0{,}005$	2.2269057	2.2422274	2.2525226	2.3063557	0.96
$t = 0{,}0008$	2.0188211	2.0238280	2.0271929	2.0447857	0.96

In [4], shows that the values of the controls in the middle points are approximated with order $\mathcal{O}(h^2)$. In fact, by considering the L^2-projection of the optimal associated adjoint states

$$\tilde{u}_1 = \Pi_{[a,b]} \left(-\frac{1}{\nu} p_{th}(u_h) \right), \tag{19}$$

where p_{th} is the solution of the adjoint equation of the problem (1), we obtain an admissible controls $\tilde{u}_1$ that increases the accuracy of the calculated control to orden h^2. In order to solve this primal-dual optimization problem to obtain the superconvergence result, we implement a primal-dual active set strategy, see for instance, [6].

The Figure shows the approximation error.

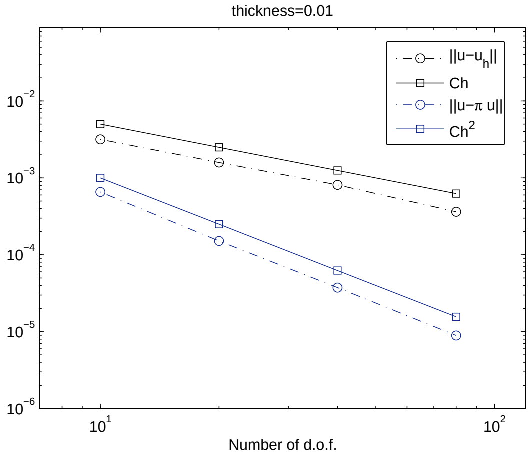

Figura 1: The superconvergence result

Distributed Sensors: The problem of AVC arises naturally from engineering problems; several of them have been reviewed in the book Fuller, Elliot and Nelson[2]. In this book they review a control scheme based on Fourier transform, applied to Euler-Bernoulli beam model. While the control scheme proposed in this paper is different from that and, moreover, the structure is modeled with different equations, Timoshenko's model, for small values of the beam thickness we can expect similar results, because the objective of the control is the same and the shear effect included in Timoshenko's equations can be ignored. To formulate this control problem is enough to consider one sensor along the beam ($a_1 = 0$ and a_1=1) and the same choices as the previous numerical examples (see [2]).

Similar to the punctual sensor test, in Table 2 we report the optimal control computed for different values of t and successively refined meshes. In this table we can see how the order of convergence remains uniformly optimal in t. In the same way as above, the extrapolated is considered as an accurate value of optimal control and it is used to compute the relative error.

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Tabla 2: Optimal controls for a distributed sensor problem (scaled by a factor 10^{10}).

	$h = 1/80$	$h = 1/100$	$h = 1/120$	ext	order
$t = 0{,}8$	-9.7823054	-9.7817365	-9.7814275	-9.7807252	2.00
$t = 0{,}1$	-8.6262707	-8.6259863	-8.6258316	-8.6254777	1.99
$t = 0{,}05$	-8.4882370	-8.4881011	-8.4880270	-8.4878565	1.98
$t = 0{,}01$	-8.4649421	-8.4649080	-8.4648892	-8.4648446	1.93
$t = 0{,}005$	-8.5598335	-8.5599232	-8.5599717	-8.5600807	2.02
$t = 0{,}0008$	-6.5088359	-6.3125978	-6.2058951	-5.9635811	2.00

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